Probability II. Solutions to Problem Sheet 5.

Part 1

1. (i)

$$\begin{aligned} P(X=x|Y=y) &= \frac{P(X=x,Y=y)}{P(Y=y)} = \frac{\frac{n!}{x!y!(n-x-y)!}p^x\theta^y(1-p-\theta)^{n-x-y}}{{}^nC_y\theta^y(1-\theta)^{(n-y)}} \\ &= {}^{n-y}C_x\frac{p^x(1-p-\theta)^{(n-y)-x}}{(1-\theta)^{n-y}} \\ &= {}^{n-y}C_x\left(\frac{p}{(1-\theta)}\right)^x\left(1-\frac{p}{(1-\theta)}\right)^{(n-y)-x} \end{aligned}$$

Hence $X|Y=y \sim Binomial\left(n-y, \frac{p}{(1-\theta)}\right)$ and so $E[X|Y]=(n-Y)\frac{p}{(1-\theta)}$. Then

$$\begin{aligned} E[XY] &= E[Y\times E[X|Y]] = \frac{p}{(1-\theta)}E[Y(n-Y)] \\ &= \frac{p}{(1-\theta)}\left(nE[Y]-(Var(Y)+(E[Y])^2)\right) = \frac{p}{(1-\theta)}\left(n^2\theta-n\theta(1-\theta)-n^2\theta^2\right) \\ &= \frac{p}{(1-\theta)}(n^2-n)\theta(1-\theta) = n(n-1)p\theta \end{aligned}$$

Hence $Cov(X,Y)=E[XY]-E[X]E[Y]=n(n-1)p\theta-n^2p\theta=-np\theta$

(ii) $Z=X+Y$. Hence $G_Z(t)=E[t^Z]=E[t^{X+Y}]=E[t^Xt^Y]=G_{X,Y}(t,t)$. But from lectures $G_{X,Y}(s,t)=(ps+\theta t+(1-p-\theta))^n$. Therefore $G_Z(t,t)=(pt+\theta t+(1-p-\theta))^n=((p+\theta)t+(1-(p+\theta)))^n$ which is the p.g.f. of binomial so that $Z\sim Binomial(n,p+\theta)$.

$$\begin{aligned} P(X=x|Z=z) &= \frac{P(X=x,Z=z)}{P(Z=z)} = \frac{P(X=x,Y=z-x)}{P(Z=z)} \\ &= \frac{\frac{n!}{x!(z-x)!(n-z)!}p^x\theta^{z-x}(1-p-\theta)^{n-z}}{{}^nC_z(p+\theta)^z(1-p-\theta)^{(n-z)}} \\ &= {}^zCx\frac{p^x\theta^{z-x}}{(p+\theta)^z} = {}^zCx\left(\frac{p}{(p+\theta)}\right)^x\left(1-\frac{p}{(p+\theta)}\right)^{z-x} \end{aligned}$$

Hence $X|Z=z\sim Binomial\left(z,\frac{p}{(p+\theta)}\right)$

Part 2

2.A. (i) $E(S_N) = aE(N), \; Var(S_N) = \sigma^2 E(N) + a^2 Var(N)$

(ii) The shop receives $S = \sum_{j=1}^{N} X_j$ pounds. Since $N \sim Poisson(100)$ we have: $E(N) = 100$, $Var(N) = 100$. Also, in our case $a = 20$, $\sigma^2 = 40$. Hence

$$E(S) = 20 \times 100 = 2000, \quad Var(S) = 40 \times 100 + 20^2 \times 100 = 44000.$$

2.B. (i) Define $Z_j = 1$ if the j^{th} message is not detected by the spam filter and $Z_j = 0$ if it is detected. Then $Z_j \sim Bernoulli(p)$ and $X = \sum_{j=1}^{Y} Z_j$. Hence, applying the above formulae gives

$$E[X] = E(Z)E(Y) = p\mu, \quad Var(X) = Var(Z)E(Y) + p^2 Var(Y) = pq\mu + p^2\mu = p\mu.$$

(We use here that $E(Y) = \mu$, $Var(Y) = \mu$ and by definition the r.v. $Z \sim Bernoulli(p)$ has the same distribution as any of Z_j; by definition $q = 1 - p$. Remember also that $E(Z) = p, \; Var(Z) = pq$.)

(ii) If $Y = y$ then $X = \sum_{j=1}^{y} Z_j$. Hence

$$E[t^X|Y = y] = E\left[\prod_{j=1}^{y} t^{Z_j}\right] = \prod_{j=1}^{y} E[t^{Z_j}] = \left(E[t^Z]\right)^y = (pt + q)^y.$$

But then $E[t^X|Y]] = (pt + q)^Y$ and

$$G_X(t) = E[E[t^X|Y]] = E[(pt + q)^Y] = G_Y(pt + q) = e^{\mu((pt+q)-1)} = e^{p\mu(t-1)}.$$

(Note that the fact that $G_Y(s) = e^{\mu(s-1)}$ is due to Y being a Poisson r.v. and is used without proof which was given a long time ago.)

(iii) But this is the p.g.f. of a Poisson r.v. with parameter $\lambda = p\mu$. Hence by the uniqueness of the p.g.f., $X \sim Poisson(p\mu)$, or, equivalently, $P(X = k) = e^{-p\mu}\frac{(p\mu)^k}{k!}$.

(iv) In this case $G_Y(t) = \sum_{k=0}^{\infty} \bar{p}\bar{q}^k t^k = \frac{\bar{p}}{1-\bar{q}t}$, $E(Y) = \frac{\bar{q}}{\bar{p}}$, $Var(Y) = \frac{\bar{q}}{\bar{p}^2}$. Hence

$$E[X] = E(Z)E(Y) = p\frac{\bar{q}}{\bar{p}}, \quad Var(X) = Var(Z)E(Y) + p^2 Var(Y) = pq\frac{\bar{q}}{\bar{p}} + p^2\frac{\bar{q}}{\bar{p}^2}.$$

To find the p.g.f. of X we note first that $E[t^X|Y = y]$ depends only on y and the distribution of Z. The calculation is therefore exactly the same as in (ii):

$$E[t^X|Y = y] = E\left[\prod_{j=1}^{y} t^{Z_j}\right] = \prod_{j=1}^{y} E[t^{Z_j}] = \left(E[t^Z]\right)^y = (pt + q)^y.$$

Hence $E[t^X|Y]] = (pt+q)^Y$ (but the distribution of this r.v. is different from the one in (ii)!). We now have

$$G_X(t) = E[E[t^X|Y]] = E[(pt+q)^Y] = G_Y(pt+q) = \frac{\bar{p}}{1-\bar{q}(pt+q)}.$$

Exercise. Prove that $X \sim Geometric(\frac{p}{1-\bar{q}q})$. (You were not required to identify the distribution of X but given the $G_X(t)$ this is a very easy thing to do.)