Probability II. Solutions to Problem Sheet 4.

1. Let X give the number of offspring for a bacterium. Then the corresponding $G(t) = \frac{1}{4} + \frac{1}{2}t^2 + \frac{1}{4}t^3$.

(a) As you will know (see lecture) $\theta_1 = G(0) = \frac{1}{4}$. From the relation $\theta_n = G(\theta_{n-1}$ we have

$$\theta_2 = \frac{1}{4} + \frac{1}{2}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{4}\right)^3 = \frac{73}{256} = 0.2852$$

and

$$\theta_3 = G(\theta_2) = \frac{1}{4} + \frac{1}{2}\left(\frac{73}{256}\right)^2 + \frac{1}{4}\left(\frac{73}{256}\right)^3 = 0.29645$$

(b) Use results from lectures with $\mu = E[X] = \frac{7}{4}$ and $\sigma^2 = E[X^2] - (E[X])^2 = \frac{17}{4} - \frac{49}{16} = \frac{19}{16}$

$$E[Y_3] = \mu^3 = \left(\frac{7}{4}\right)^3 = \frac{343}{64} = 5.359375$$

$$\begin{aligned} Var(Y_3) &= \frac{\sigma^2\mu^2\left(1-\mu^3\right)}{(1-\mu)} \\ &= \frac{\frac{19}{16} \times \frac{49}{16} \times 4.359375}{\frac{3}{4}} \\ &= \frac{19 \times 49 \times 4.359375}{192} = 21.13843 \end{aligned}$$

(c) From lectures, θ is the smallest positive root of $G(t) = t$. So we solve $\frac{1}{4} + \frac{1}{2}t^2 + \frac{1}{4}t^3 = t$ i.e. $t^3 + 2t^2 - 4t + 1 = 0$, i.e. $(t-1)(t^2+3t-1) = 0$. The roots are $t_1 = 1$ and $t_2 = \frac{-3+\sqrt{13}}{2}$, $t_2 = \frac{-3-\sqrt{13}}{2}$. Hence $\theta = \frac{-3+\sqrt{13}}{2}$.

(d) If $Y_0 = 3$ then we find the mean and variance for the number of bacteria in generation 3 by multiplying the previous results by 3. So now $E[Y_3] = 3 \times \frac{343}{64} = 16.078125$ and $Var(Y_3) = 3 \times 21.13843 = 63.41529$

Also we find the probability of eventual extinction by taking the value in part (d) to the power 3, i.e. the probability of eventual extinction is $\left(\frac{-3+\sqrt{13}}{2}\right)^3$.

2. Let Y count the number of children for a male.

Either: use generating functions. Now $G_Y(t) = \frac{1}{4} + \frac{3}{4}t^2$. Also $X|Y = y \sim Binomial\left(y, \frac{3}{4}\right)$ so that $E[t^X|Y = y] = \left(\frac{1}{4} + \frac{3}{4}t\right)^y$. Hence

$$\begin{aligned} G_X(t) &= E[E[t^X|Y]] = E\left[\left(\frac{1}{4} + \frac{3}{4}t\right)^Y\right] = G_Y\left(\frac{1}{4} + \frac{3}{4}t\right) \\ &= \frac{1}{4} + \frac{3}{4}\left(\frac{1}{4} + \frac{3}{4}t\right)^2 = \frac{19}{64} + \frac{18}{64}t + \frac{27}{64}t^2 \end{aligned}$$

Or: Use theorem of total probability $P(X = x) = P(X = x|Y = 0)P(Y = 0) + P(X = x|Y = 2)P(Y = 2)$. Since $P(X = 0|Y = 0) = 1$ and $P(X = x|Y = 0) = 0$ if $x = 1, 2$, $P(X = 0) = \frac{1}{4} + \left(\frac{1}{4}\right)^2 \frac{3}{4} = \frac{19}{64}$, $P(X = 1) = \left(2\left(\frac{1}{4}\right)\left(\frac{3}{4}\right)\right)\frac{3}{4} = \frac{18}{64}$ and $P(X = 2) = \left(\frac{3}{4}\right)^2 \frac{3}{4} = \frac{27}{64}$ Hence $G_X(t) = \frac{19}{64} + \frac{18}{64}t + \frac{27}{64}t^2$

We first find $\theta =$ P(eventual extinction) when there is just one ancestor called Ramsbottom. Then θ is the smallest positive root of $t = G_X(t)$, i.e. of $t = \frac{19}{64} + \frac{18}{64}t + \frac{27}{64}t^2$, i.e. $27t^2 - 46t + 19 = 0$, i.e. $(t - 1)(27t - 19) = 0$. Hence $\theta = \frac{19}{27}$

(i) If there are k boys called Ramsbottom initially, then the probability of eventual extinction is just $\theta^k = \left(\frac{19}{27}\right)^k$.

(ii) Using the theorem of total probability, if we let E be the event 'the name Ramsbottom eventually dies out',

$$\begin{aligned} P(E) &= \sum_{k=0}^{N} P(E|K = k)P(K = k) = \sum_{k=0}^{N} \theta^k P(K = k) = G_K(\theta) \\ &= \left(\frac{1}{2} + \frac{1}{2}\theta\right)^N = \left(\frac{1}{2} + \frac{1}{2}\frac{19}{27}\right)^N = \left(\frac{23}{27}\right)^N \end{aligned}$$