Probability II. Solutions to Problem Sheet 3.

1. Let X give the number of offspring for a bacterium. Then $G_X(t) = \frac{2}{5} + \frac{3}{5}t^2$.

(a)

$$\begin{aligned} G_{Y_2}(t) &= G_{Y_1}(G_X(t)) = G_X(G_X(t)) = \frac{2}{5} + \frac{3}{5}\left(\frac{2}{5} + \frac{3}{5}t^2\right)^2 \\ &= \frac{2}{5} + \frac{3}{5}\left(\frac{4}{25} + \frac{12}{25}t^2 + \frac{9}{25}t^4\right) = \frac{62}{125} + \frac{36}{125}t^2 + \frac{27}{125}t^4 \end{aligned}$$

Therefore $P(Y_2 = 0) = \frac{62}{125}$, $P(Y_2 = 2) = \frac{36}{125}$ and $P(Y_2 = 4) = \frac{27}{125}$.

(b) Either

$$\theta_3 = G_X(G_{Y_2}(0)) = G_X\left(\frac{62}{125}\right) = \frac{2}{5} + \frac{3}{5}\left(\frac{62}{125}\right)^2 = \frac{42782}{78125} = 0.5476096$$

or

$$\theta_3 = G_{Y_2}(G_X(0)) = G_{Y_2}\left(\frac{2}{5}\right) = \frac{2}{5} + \frac{3}{5}\left(\frac{2}{5} + \frac{3}{5} \times \frac{4}{25}\right)^2 = \frac{2}{5} + \frac{3}{5}\left(\frac{62}{125}\right)^2 = 0.5476096$$

(c) Use results from lectures with $\mu = E[X] = \frac{6}{5}$ and $\sigma^2 = E[X^2] - (E[X])^2 = \frac{12}{5} - \frac{36}{25} = \frac{24}{25}$

$$E[Y_3] = \mu^3 = \left(\frac{6}{5}\right)^3 = \frac{216}{125} = 1.728$$

$$\begin{aligned} Var(Y_3) &= \frac{\sigma^2\mu^2\left(1-\mu^3\right)}{(1-\mu)} \\ &= \frac{\frac{24}{25} \times \frac{36}{25}\left(\frac{91}{125}\right)}{\frac{1}{5}} \\ &= \frac{78624}{15625} = 5.031936 \end{aligned}$$

(d) From lectures, θ is the smallest positive root of $G_X(t) = t$. So we solve $\frac{2}{5} + \frac{3}{5}t^2 = t$ i.e. $3t^2 - 5t + 2 = 0$, i.e. $(t-1)(3t-2) = 0$. The roots are $t = 1$ and $t = \frac{2}{3}$. Hence $\theta = \frac{2}{3}$.

(e) If $Y_0 = 5$ then we find the mean and variance for the number of bacteria in generation 3 by multiplying the previous results by 5. So now $E[Y_3] = 5 \times \frac{216}{125} = \frac{216}{25} = 8.64$ and $Var(Y_3) = 5 \times \frac{78624}{15625} = \frac{78624}{3125} = 25.15968$

Also we find the probability of eventual extinction by taking the value in part (d) to the power 5, i.e. the probability of eventual extinction is $\left(\frac{2}{3}\right)^5 = \frac{32}{243} = 0.131687$

2.

$$G_X(t) = \sum_{x=0}^{\infty} \left(\frac{1}{2}\right)^{x+1} t^x = \sum_{x=0}^{\infty} \frac{1}{2}\left(\frac{t}{2}\right)^x = \frac{\frac{1}{2}}{\left(1 - \frac{t}{2}\right)} = \frac{1}{2-t}$$

We prove that $\theta_n = P(Y_n = 0) = \frac{n}{n+1}$ by induction on n.

The result holds for $n = 0$ since $Y_0 \equiv 1$ and hence $\theta_0 = P(Y_0 = 0) = 0$ which is just $\frac{0}{0+1}$.

Now assume that $Y_n = \frac{n}{n+1}$ for all non-negative integers $n \leq N$. We now show that it holds for $n = N + 1$.

From lectures $\theta_{N+1} = G_X(\theta_N)$. Also from the inductive hypothesis with $n = N$, $\theta_N = \frac{N}{N+1}$. Therefore

$$\theta_{N+1} = G_X(\theta_N) = \frac{1}{2 - \theta_N} = \frac{1}{2 - \frac{N}{N+1}} = \frac{N+1}{2(N+1) - N} = \frac{N+1}{(N+1)+1}$$

Hence the inductive hypothesis also holds for $n = N+1$. Hence by induction $\theta_n = \frac{n}{n+1}$ for all non-negative integers n.

The probability of eventual extinction of the branching process is $\theta = \lim_{n\to\infty} \theta_n = \lim_{n\to\infty} \frac{n}{n+1} = 1$. So the branching process is certain to eventually become extinct. the lecture **3.** This problem can be solved in the same way as 4.A. below. It is also similar to what we discussed in Lecture 9.

4.A. Substituting the expression for z_k into the left hand side of the main equation yields

$$\begin{aligned}&az_{k+1} + bz_k + az_{k-1} = a(c_1 + c_2(k+1) + c_3(k+1)^2) +\\ &b(c_1 + c_2k + c_3k^2) + a(c_1 + c_2(k-1) + c_3(k-1)^2) =\\ &c_1(2a+b) + c_2(2a+b)k + c_3((2a+b)k^2 + 2a) \equiv 2ac_3\end{aligned}$$

We thus have $2ac_3 = f$ and hence $c_3 = \frac{f}{2a}$.

The equations $z_M = 0$ and $z_N = 0$ now imply

$$\begin{cases} c_1 + c_2 M + c_3 M^2 = d \\ c_1 + c_2 N + c_3 N^2 = 0 \end{cases}$$

Hence $c_2(N - M) + c_3(N^2 - M^2) = -d$ and

$$c_2 = \frac{-d}{N - M} - c_3(N + M) = \frac{-d}{N - M} - \frac{f}{2a}(N + M).$$

Finally

$$c_1 = -c_2 N - c_3 N^2 = \frac{dN}{N - M} + \frac{f}{2a} MN$$

and

$$z_k = \frac{dN}{N - M} + \frac{f}{2a} MN + (\frac{-d}{N - M} - \frac{f}{2a}(N + M))k + \frac{f}{2a} k^2.$$

4.B. Let B_1, B_2 and B_3 be the events that Joe wins, draws or loses the first game.

If A_k is the event that Joe loses his money starting from k units and $L_k = P(A_k)$, from the theorem of total probability

$$L_k = P(A_k) = P(A_k|B_1)P(B_1)+P(A_k|B_2)P(B_2)+P(A_k|B_3)P(B_3) = L_{k+1}\frac{1}{4}+L_k\frac{1}{2}+L_{k-1}\frac{1}{4}$$

Hence $L_{k+1} - 2L_k + L_{k-1} = 0$ for $k = 1, ..., N - 1$. Also $L_0 = 1$ and $L_N = 0$. These equations can be viewed as a particular case of those considered in 4.A with $f = 0$, $d = 1$, $a = \frac{1}{4}$. The solution is therefore

$$L_k = c_1 + c_2 k = 1 - \frac{k}{N} = \frac{N - k}{N}.$$

So the probability he loses all his money is $\frac{N-k}{N}$.

Note that you could have obtained this result from the ordinary gambler's ruin result by only considering games where he does not draw. The probability p will then be the probability of winning given that the game is not drawn (so is 1/2 here).

Let T_k be the number of games he plays starting from k units and let $E_k = E[T_k]$.

$$\begin{aligned} E_k &= E[T_k] = E[T_k|B_1]P(B_1) + E[T_k|B_2]P(B_2) + E[T_k|B_3]P(B_3) \\ &= \frac{1}{4}(1 + E_{k+1}) + \frac{1}{2}(1 + E_k) + \frac{1}{4}(1 + E_{k-1}) \end{aligned}$$

Re-arranging gives $E_{k+1} - 2E_k + E_{k-1} = -4$. Also $E_0 = E_N = 0$. Once again, this is a particular case of 4.A with $d = 0$, $f = -4$, $a = 1$, $M = 0$. Hence

$$E_k = c_1 + c_2 k + c_3 k^2 = 2Mk - 2k^2 = 2k(M - k).$$

Therefore the expected duration of the game starting from k units is $E_k = 2k(N - k)$