Probability II. Solutions to Problem Sheet 2.

1. This is just the gambler's ruin problem with $p = \frac{18}{37}$. We want $r_{500}(0, 1000)$. From notes

$$r_k(M,N) = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^M}{\left(\frac{q}{p}\right)^N - \left(\frac{q}{p}\right)^M}$$

Therefore

$$r_{500}(0,1000) = \frac{\left(\frac{19}{18}\right)^{500} - 1}{\left(\frac{19}{18}\right)^{1000} - 1} = \frac{1}{\left(\frac{19}{18}\right)^{500} + 1} = 1.8174 \times 10^{-12}$$

If Kermit's stake at each game is £10 we change units and write each of k, M and N in terms of new units. So we want to find

$$r_{50}(0,100) = \frac{\left(\frac{19}{18}\right)^{50} - 1}{\left(\frac{19}{18}\right)^{100} - 1} = \frac{1}{\left(\frac{19}{18}\right)^{50} + 1} = 0.062775$$

2. Let X be the amount he wins. Let B_1, be the event 'the first throw is 6', B_2 be the event 'the first throw is a 1 and B_3 be the event 'the first throw is $2, 3, 4$ or 5.

$$\begin{aligned} E[X] &= E[X|B_1]P(B_1) + E[X|B_2]P(B_2) + E[X|B_3]P(B_3) \\ &= (C - 1 + E[X])\frac{1}{6} + (-1)\frac{1}{6} + (-1 + E[X])\frac{2}{3} \\ &= \frac{C-6}{6} + \frac{5}{6}E[X] \end{aligned}$$

Hence $E[X] = C - 6$. the game is fair if $E[X] = 0$, i.e. $C = 6$.

3.A. Let B_1 be the event that the first throw gives 2 sixes, B_2 be the event that the first throw gives a product of 6 and B_3 be the event that the first throw has a product which is not 6 or 36. Then

$P(B_1) = \frac{1}{36}$, $P(B_2) = \frac{4}{36} = \frac{1}{9}$ and $P(B_3) = \frac{31}{36}$.

$$P(E) = P(B_1)P(E|B_1) + P(B_2)P(E|B_2) + P(B_3)P(E|B_3) = \frac{1}{36} \times 0 + \frac{1}{9} \times 1 + \frac{31}{36} \times P(E)$$

Therefore $\frac{5}{36}P(E) = \frac{1}{9}$ and so $P(E) = \frac{4}{5}$.

Remark. You can also see this by symmetry. There are 5 values on the pair of dice for which the game ends, namely $\{(6,6),(1,6),(6,1),(2,3),(3,2)\}$. The game is equally likely to end on each of these. Four correspond to a product of 6. Therefore $P(E) = \frac{4}{5}$.

3.B. This is equivalent to the gambler's ruin problem with $p = 1/2$, $k = 3$, $M = 0$, $N = 9$ where we want $r_3(0,9)$.

When $p = 1/2$, $r_k(M,N) = \frac{k-M}{N-M}$, and hence $r_3(0,9) = \frac{3-0}{9-0} = \frac{1}{3}$.

3.C. Here $p = 1/2$ so that $E_k(M,N) = (k-M)(N-k)$. We want $E_{50}(0,100) = (50-0)(100-50) = 2500$

If his stake at each game is £5 we simply alter the units so £5 is one new unit. Then in new units $k = 10$, $M = 0$ and $N = 20$. Hence we want $E_{10}(0,20) = (10-0)(20-10) = 100$.