Probability II. Solutions to Sheet 10.

Part 1

1. Let $\mathbf{X}$ have vector of means μ and variance-covariance matrix $\mathbf{V}$ and let $\mathbf{Y} = \mathbf{AX}$. then

$$\mu = \begin{pmatrix} 2 \\ 5 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}.$$

Therefore

$$E[\mathbf{Y}] = \mathbf{A}\mu = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 7 \\ -13 \end{pmatrix}$$

and the variance-covariance matrix for $\mathbf{Y}$ is

$$\mathbf{AVA}^T = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 3 & 7 \\ 1 & -7 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 28 \end{pmatrix}$$

Y_1 and Y_2 are linear functions of bivariate normals so are bivariate normal. Since $Cov(Y_1, Y_2) = 0$, in fact Y_1 and Y_2 are independent normal. Therefore $Y_1 \sim N(7, 4)$ independent of $Y_2 \sim N(-13, 28)$

2. Since $Y|X = x$ has normal distribution with mean $\alpha + \beta x$ and variance σ^2, then $E\left[e^{tY}|X = x\right] = e^{(\alpha+\beta x)t+\frac{1}{2}\sigma^2t^2}$. Also $X \sim N(\eta, \tau^2)$. Therefore $M_X(t) = e^{\eta t+\frac{1}{2}\tau^2t^2}$. Hence

$$\begin{aligned} M_Y(t) &= E[e^{tY}] = E[E[e^{tY}|X]] = E\left[e^{(\alpha+\beta X)t+\frac{1}{2}\sigma^2t^2}\right] \\ &= e^{\alpha t+\frac{1}{2}\sigma^2t^2}M_X(\beta t) \\ &= e^{\alpha t+\frac{1}{2}\sigma^2t^2}e^{\eta(\beta t)+\frac{1}{2}(\beta t)^2\tau^2} \\ &= e^{(\alpha+\beta\eta)t+\frac{1}{2}(\sigma^2+\beta^2\tau^2)t^2} \end{aligned}$$

This is the m.g.f. of a normal distribution, hence $Y \sim N(\alpha + \beta\eta, \sigma^2 + \beta^2\tau^2)$.

The joint m.g.f. for X and Y is

$$
\begin{aligned}
M_{X,Y}(s,t) &= E[E[e^{sX+tY}|X]] = E[e^{sX}E[e^{tY}|X]] \\
&= E\left[e^{sX}e^{(\alpha+\beta X)t+\frac{1}{2}\sigma^2t^2}\right] \\
&= e^{\alpha t+\frac{1}{2}\sigma^2t^2}M_X(s+\beta t) \\
&= e^{\alpha t+\frac{1}{2}\sigma^2t^2}e^{\eta(s+\beta t)+\frac{1}{2}\tau^2(s+\beta t)^2} \\
&= e^{(\eta s+(\alpha+\beta\eta)t)+\frac{1}{2}\left(\tau^2s^2+(\sigma^2+\beta^2\tau^2)t^2+2st\beta\tau^2\right)}
\end{aligned}
$$

This is the joint m.g.f. of a bivariate normal distribution. So X and Y have bivariate normal distribution with means η and $\alpha+\beta\eta$, variances τ^2 and $\sigma^2+\beta^2\tau^2$ and covariance $\beta\tau^2$ (or equivalently coefficient of correlation $\frac{\beta\tau^2}{\sqrt{\tau^2(\sigma^2+\beta^2\tau^2)}}$).

Part 2

3. (a) Markov's inequality for a non-negative r.v. X with mean μ states that for any $h>0$, $P(X \geq h) \leq \frac{\mu}{h}$. So here we simply take $h=\mu+2\sigma$ to obtain

$$P(X \geq \mu+2\sigma) \leq \frac{\mu}{\mu+2\sigma}$$

So the upper bound for $P(X \geq \mu+2\sigma)$ is $\frac{\mu}{\mu+2\sigma}$.

(b) If X has mean μ and variance σ^2 then Chebyshev's inequality states that, for any $h>0$,

$$P(|X-\mu| \geq h) \leq \frac{\sigma^2}{h^2}$$

So we just need to take $h=2\sigma$. Then Chebyshev's inequality states that

$$P(|X-\mu)| \geq 2\sigma) \leq \frac{\sigma^2}{(2\sigma)^2} = \frac{1}{4}$$

So the upper bound for $P(|X-\mu| \geq 2\sigma)$ is $\frac{1}{4}$.

If $X \sim Exp(\theta)$, then $\mu=\frac{1}{\theta}$ and $\sigma^2=\frac{1}{\theta^2}$. Then:

(a) Markov's inequality is just $P\left(X \geq \frac{3}{\theta}\right) \leq \frac{1}{3}$. The exact probability is just

$$P\left(X \geq \frac{3}{\theta}\right) = \int_{\frac{3}{\theta}}^{\infty} \theta e^{-\theta x} dx = e^{-3} = 0.04979$$

(b) Chebyshev's inequality is just $P\left(\left|X - \frac{1}{\theta}\right| \geq \frac{2}{\theta}\right) \leq \frac{1}{4}$ The exact probability is just

$$P\left(\left|X - \frac{1}{\theta}\right| \geq \frac{2}{\theta}\right) = P\left(X \geq \frac{3}{\theta}\right) + P\left(X \leq -\frac{1}{\theta}\right) = \int_{\frac{3}{\theta}}^{\infty} \theta e^{-\theta x} dx = e^{-3} = 0.04979$$

4. $E[\overline{X}_n] = p$ and $Var(\overline{X}_n) = \frac{p(1-p)}{n}$. Applying Chebyshev's inequality to $\overline{X}_n$, and letting $h = 0.1p$ gives

$$P(|\overline{X}_n - p| \geq 0.1p) \leq \frac{p(1-p)/n}{(0.1p)^2} = \frac{100(1-p)}{np}$$

Hence $P(|\overline{X}_n - p| \geq 0.1p) \leq 0.05$ provided $\frac{100(1-p)}{np} \leq 0.05$, i.e. $n \geq \frac{100(1-p)}{0.05p} = \frac{2000(1-p)}{p}$.

The Central Limit Theorem implies that if $Z = \frac{\sqrt{n}(\overline{X}_n - p)}{\sqrt{p(1-p)}}$, then for n large $P(Z \leq z) \simeq \Phi(z)$ where Φ is the c.d.f. for the $N(0,1)$ distribution. Here we want

$$0.05 = P(|\overline{X}_n - p| \geq 0.1p) = P\left(|Z| \geq \frac{\sqrt{n}0.1p}{\sqrt{p(1-p)}}\right) \simeq 2\left(1 - \Phi\left(0.1\sqrt{\frac{np}{(1-p)}}\right)\right)$$

Hence $\Phi\left(0.1\sqrt{\frac{np}{(1-p)}}\right) \simeq 0.975$, so $0.1\sqrt{\frac{np}{(1-p)}} = 1.96$ and therefore $n \approxeq \frac{(19.6)^2(1-p)}{p} = 384.16\left(\frac{1}{p} - 1\right)$.

Any value of n greater than this value will give a smaller probability for $P(|\overline{X}_n - p| \geq 0.1p)$ than 0.05.

Note that $\frac{1}{p}$ is largest when p is smallest, so that we require the largest sample size for the smallest value of p. When $p = 0.25$ then the sample size required so that $P(|\overline{X}_n - p| \geq 0.1p) \simeq 0.05$ is approximately $384.16 \times 3 = 1152.48$.

Therefore the smallest sample size N required so that $P(|\overline{X}_n - p| \geq 0.1p) \leq 0.05$ for all $0.25 \leq p \leq 0.75$ and all $n \geq N$ is 1153.