Probability II. Solutions to Problem Sheet 1.

Part 1

1. (a) If $G_X(t)$ is a p.g.f. then $G_X(1) = 1$. Here $G_X(1) = \frac{3}{2}$. So $G_X(t)$ cannot be a p.g.f.

(b) $G_X(t) = \frac{1}{4}(1+t^2)^2$. $G_X(1) = 1$ and $G_X(t) = \frac{1}{4} + \frac{1}{2}t^2 + \frac{1}{4}t^4$, so all coefficients in the power series expansion are non-negative (with coefficients which sum to one since $G_X(1) = 1$. Hence it is a p.g.f and the corresponding probability mass function has $P(X=0) = \frac{1}{4}$, $P(X=2) = \frac{1}{2}$, $P(X=4) = \frac{1}{4}$ and $P(X=x) = 0$ for all other non-negative integer values of X.

(c) $G_X(t) = \frac{2}{(1+t)}$. Although $G_X(1) = 1$, $G_X(t)$ is not a p.g.f.

You can show this by noting that $G_X(0) = 2$ so that if $G_X(t)$ were a p.g.f., $P(X = 0) = 2$, which is impossible.

Alternatively you can note that $G_X(t)$ is not an increasing function of t for $t \geq 0$ (in fact $G'_X(t) < 0$ for $t \geq 0$) so $G_X(t)$ cannot be a p.g.f.

You could also look at the power series expansion, which is

$$G_X(t) = 2 - 2t + 2t^2 - 2t^3 + ... = \sum_{x=0}^{\infty} 2(-1)^x t^x$$

Odd powers of t have negative coefficients and all probabilities must be non-negative, so $G_X(t)$ is not a p.g.f.

2. (a) $G_X(t) = \left(\frac{3}{4} + \frac{1}{4}t\right)^3$. This is the p.g.f. corresponding to a binomial distribution with $n = 3$ and $p = \frac{1}{4}$. Hence $X \sim \mathit{Binomial}(3, \frac{1}{4})$.

(b) Note that $G_X(t) = \frac{1}{64}(3+t)^3 = \left(\frac{3}{4} + \frac{1}{4}t\right)^3$, which is identical to the p.g.f. in part (a). Hence $X \sim \mathit{Binomial}(3, \frac{1}{4})$.

(c) $G_X(t) = e^{5t-5} = e^{5(t-1)}$. This is the p.g.f. corresponding to a Poisson distribution with $\lambda = 5$.

3. Let X be a random variable with probability generating function

$G_X(t) = \frac{t}{3-2t} = \frac{\frac{1}{3}t}{1-\frac{2}{3}t}$. This is the p.g.f. corresponding to a geometric distribution with $p = \frac{1}{3}$. Hence $X \sim \mathit{Geometeric}(\frac{1}{3})$.

We can differentiate $G_X(t)$ to obtain $P(X=1)$, $P(X=2)$, $E(X)$ and $\mathrm{Var}(X)$.

$$G_X^{'}(t) = \frac{1}{3-2t} + \frac{2t}{(3-2t)^2}$$

Hence $E[X] = G_X^{'}(1) = 1 + 2 = 3$ and $P(X=1) = G_X^{'}(0) = \frac{1}{3}$.

$$G_X^{''}(t) = \frac{2}{(3-2t)^2} + \frac{2}{(3-2t)^2} + \frac{8t}{(3-2t)^3}$$

Hence $E[X(X-1)] = G_X^{''}(1) = 2 + 2 + 8 = 12$ and so

$$Var(X) = E[X(X-1)] + E[X] - (E[X])^2 = 12 + 3 - 9 = 6$$

Also $P(X=2) = \frac{1}{2}G_X^{''}(0) = \frac{1}{2}\left(\frac{2}{9} + \frac{2}{9}\right) = \frac{2}{9}$.

4. $G_X(t) = \left(\frac{1}{2}t + \frac{1}{2}\right)$ and $G_Y(t) = \left(\frac{2}{3}t + \frac{1}{3}\right)$. Hence

$$G_Z(t) = G_X(t)G_Y(t) = \left(\frac{1}{2}t + \frac{1}{2}\right)\left(\frac{2}{3}t + \frac{1}{3}\right) = \frac{1}{6} + \frac{1}{2}t + \frac{1}{3}t^2$$

Therefore $P(Z=0) = \frac{1}{6}$, $P(Z=1) = \frac{1}{2}$ and $P(Z=2) = \frac{1}{3}$. ($P(Z=z) = 0$ for all other non-negative values of Z.)

Part 2

5.A. Since $G_X(t) = e^{\lambda(t-1)}$ and $G_Y(t) = e^{\mu(t-1)}$, we obtain

$$G_Z(t) = G_X(t)G_Y(t) = e^{\lambda(t-1)}e^{\mu(t-1)} = e^{(\lambda+\mu)(t-1)}$$

Since this is the p.g.f. corresponding to a Poisson distribution with parameter $(\lambda+\mu)$, we see that $Z \sim Poisson(\lambda+\mu)$.

Similarly, when $W = \sum_{j=1}^{n} X_j$, we obtain

$$G_W(t) = \prod_{j=1}^{n} G_{X_j}(t) = \prod_{j=1}^{n} e^{\lambda(t-1)} = e^{n\lambda(t-1)}$$

Hence $W \sim Poisson(n\lambda)$.

5.B. We differentiate $G_X(t) = \frac{(t+t^2)}{2(3-2t)}$ to obtain $E(X)$ and $E[X(X-1)]$.

$$G_X^{'}(t) = \frac{(1+2t)}{2(3-2t)} + \frac{(t+t^2)}{(3-2t)^2}$$

Therefore $E[X] = G_X^{'}(1) = \frac{3}{2} + 2 = \frac{7}{2}$. Next,

$$G_X^{''}(t) = \frac{1}{(3-2t)} + \frac{(1+2t)}{(3-2t)^2} + \frac{(1+2t)}{(3-2t)^2} + \frac{4(t+t^2)}{(3-2t)^3}$$

Therefore $E[X(X-1)] = G_X^{''}(1) = 1 + 63 + 3 + 8 = 15$ and hence $Var(X) = E[X(X-1)] + E[X] - (E[X])^2 = 15 + \frac{7}{2} - \frac{49}{4} = \frac{25}{4}$

5.C. We know that a Bernoulli(p) r. v. has p. g. f. of the form $q + pt$. Since $G_X(t) = \frac{(t+t^2)}{2(3-2t)} = (t+1) \times \frac{t}{2(3-2t)}$ it is reasonable to conjecture that one of the random variables Y, Z is Bernoulli. Let us suppose that this is Y; then its p.g.f. must be given by $G_Y(t) = c(1+t)$, where c is a constant such that $G_Y(1) = 1$, that is $c(1+1) = 1$ and hence $c = \frac{1}{2}$. We thus see that $G_Y(t) = \frac{1}{2}(1+t) = \left(\frac{1}{2} + \frac{1}{2}t\right)$ in which case we must have

$$G_Z(t) = \frac{G_X(t)}{G_Y(t)} = \frac{\frac{1}{3}t}{1 - \frac{2}{3}t}.$$

Hence $Y \sim Bernoulli\left(\frac{1}{2}\right)$ and $Z \sim Geometric\left(\frac{1}{3}\right)$.

Then $E[X] = E[Y] + E[Z] = \frac{1}{2} + 3 = \frac{7}{2}$ and $Var(X) = Var(Y) + Var(Z) = \frac{1}{4} + 6 = \frac{25}{4}$.

Remark. Note that you can interchange the roles of Y and Z, i. e. the other correct answer is $Z \sim Bernoulli\left(\frac{1}{2}\right)$ and $Y \sim Geometric\left(\frac{1}{3}\right)$.