

MTH5118 Probability II. Problem Sheet 9.

This coursework consists of two parts. You are required to submit solutions to the second part only. You are strongly encouraged to solve all problems on this problem sheet. Please staple your coursework and post it in the Blue Box in the basement of the Maths building by 10:30 on Thursday, 10 December 2009.

Part 1

1. A device has two components whose random lifetimes are independent. One of the components operates initially. When this component fails the second one comes into use. Let X and Y be the lifetimes of the first (initial) and second components, where $X \sim \text{Gamma}(\theta, 2)$ independent of $Y \sim \text{Gamma}(\theta, 2)$.

Let $U = X$ and $V = X + Y$, so that $X + Y$ measures the time until the device fails. Find the joint p.d.f. for U and V .

Obtain the marginal p.d.f. for V and hence obtain the conditional p.d.f. for $U|V = v$.

2. $X \sim \text{Exp}(1)$ independent of $Y \sim \text{Exp}(1)$. Let $U = \frac{X}{Y}$ and $V = X + Y$. Find the joint p.d.f. for U and V . Hence show that U and V are independent and state their marginal p.d.f.'s.

Part 2. Please read the last two pages of Notes 10 before you try to solve the problems below.

3. Let $X|(Y = y) \sim \text{Exp}(y)$ where $Y \sim \text{Gamma}(\theta, 3)$. State the value of $E[X|Y = y]$ 10
and $\text{Var}(X|Y = y)$. Hence find $E[X]$ and $\text{Var}(X)$.

Find the joint p.d.f. for X and Y and hence find the marginal p.d.f. for X . 25

Obtain $E[X + \theta]$ and $E[(X + \theta)^2]$ and use these to check the values you have already 10
obtained for $E[X]$ and $\text{Var}(X)$.

Hint. You are reminded that $E[g(Y)|X = x] = \int_{-\infty}^{\infty} g(y)f_{Y|X}(y|x)dy$. As in the discrete case $E[g(Y)] = E[E[g(Y)|X]]$ (in particular $E[Y] = E[E[Y|X]]$) and $\text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}(E[Y|X])$.

4. The joint moment generation function of two r.v.'s X, Y is defined by $M_{X,Y}(s, t) = E[e^{sX+tY}]$. Use the properties of the m.g.f. listed in the relevant part of Notes 10 to solve the following questions.

X and Y have joint m.g.f. $M_{X,Y}(s, t) = \frac{1}{(1-s)^2-t^2}$.

Show that the marginal distribution of X is Gamma and the marginal distribution of Y is double exponential and state the parameters. 15

Find the coefficient of correlation $\rho(X, Y)$. 20

Let $U = X + Y$ and $V = X - Y$. Find the joint m.g.f. of U and V and hence show that U and V are independent and state the marginal m.g.f.'s. State the corresponding distributions. 20