MTH5118 Probability II. Problem Sheet 3.

This coursework consists of two parts. **You are required to submit solutions to the second part only. However, you are strongly encouraged to solve all problems on this problem sheet.** *Please staple your coursework and post it in the Blue Box in the basement of the Maths building by* **10:30** *on Thursday, 22nd October 2009.*

Part 1

1. A population of bacteria begins with a single individual (forming generation 0). In each generation, each individual dies with probability 2/5 or doubles (splits in two) with probability 3/5. Let Y_n be the number of bacteria in generation n.

(a) Find the p.g.f., and hence obtain the p.m.f., for Y_2.

(b) Find the probability, θ_3 that the population will die out by generation 3.

(c) Find $E[Y_3]$ and $Var(Y_3)$.

(d) Find the probability θ that the population will eventually die out.

(e) Obtain the results in (c) and (d) if initially there are 5 bacteria.

2. Consider a branching process consisting of females where Y_n is the number of females in generation n. The number of female offspring, X, of a female has p.m.f. $P(X = x) = \left(\frac{1}{2}\right)^{x+1}$ for $x = 0, 1, 2, ...$ Find $G_X(t)$. If $Y_0 \equiv 1$, prove that $\theta_n = P(Y_n = 0) = \frac{n}{n+1}$ for all non-negative integers n. Hence find the probability of eventual extinction of the branching process.

3. Consider the following system of equations:

$$\begin{cases} az_{k+1} + bz_k + cz_{k-1} = f & \text{if } M+1 \le k \le N-1 \\ z_M = d, \quad z_N = 0 & \end{cases}$$

Suppose that $a + b + c = 0$, $a \neq c$, and that $c \neq 0$.

Check that z_k can be found in the following form: $z_k = c_1 + c_2 \left(\frac{a}{c}\right)^k + c_3 k$. To this end, substitute y_k into the main equation and show that it will be satisfied for any c_1 and c_2 (which do not depend on k), whereas c_3 must be equal to $\frac{f}{a-c}$. Next, find c_1 and c_2 from the conditions $z_M = d$ and $z_N = 0$.

Part 2

4.A. Consider the following system of equations: 40

$$\begin{cases} az_{k+1} + bz_k + az_{k-1} = f & \text{if } M+1 \le k \le N-1 \\ z_M = d, \quad z_N = 0 \end{cases}$$

Suppose that $2a + b = 0$ and that $a \neq 0$.

Check that in this case z_k can be found in the following form: $z_k = c_1 + c_2 k + c_3 k^2$. To this end, substitute z_k into the main equation and show that it will be satisfied for any c_1 and c_2 (which do not depend on k), whereas c_3 has to satisfy a simple equation which defines it in a unique way. Next, find c_1 and c_2 from the conditions $z_M = d$ and $z_N = 0$.

4.B. Joe plays a series of games. At each game Joe wins, loses or draws with probabilities $\frac{1}{4}$, $\frac{1}{4}$ and $\frac{1}{2}$ respectively. He bets one unit each time and receives an additional unit, gets the unit back, or loses the unit depending upon whether he wins, draws or loses the game. He starts with k units and stops playing when he reaches 0 or N units $(0 \le k \le N)$.

Obtain the difference equations for, and hence find, L_k the probability that he goes 25
broke starting from k units.

Let T_k count the number of games he plays if he starts with k units, and let $E_k = E[T_k]$. 35
Obtain the difference equations for, and hence find, E_k.

Hint. Use the result obtained in **4.A.**