MTH5118 Probability II. Problem Sheet 10.

This coursework consists of two parts. ***You are required to submit solutions to the second part only. You are strongly encouraged to solve all problems on this problem sheet.*** *Please staple your coursework and post it in the Blue Box in the basement of the Maths building by* **10:30** *on Thursday, 17 December 2009.*

Part 1

1. Let $\mathbf{X}$ be a vector of random variables with entries X_1 and X_2, where $E[X_1] = 2$, $E[X_2] = 5$, $Var(X_1) = 4$, $Var(X_2) = 2$ and $Cov(X_1, X_2) = -1$. Write down the vector of means and the variance-covariance matrix for $\mathbf{X}$. Let $Y_1 = X_1 + X_2$ and $Y_2 = X_1 - 3X_2$ and let $\mathbf{Y}$ be the vector with entries Y_1 and Y_2. Calculate the vector of means and the variance-covariance matrix for $\mathbf{Y}$. If X_1 and X_2 have bivariate normal distribution, state the joint distribution of Y_1 and Y_2.

2. $Y|X = x \sim N(\alpha + \beta x, \sigma^2)$ and $X \sim N(\eta, \tau^2)$. Find the m.g.f. for Y and hence state the distribution of Y.

Use the result that $M_{X,Y}(s,t) = E[E[e^{sX+tY}|X]] = E[e^{sX}E[e^{tY}|X]]$ to find the joint m.g.f. for X and Y. Hence state their joint distribution.

Part 2

3. X takes non-negative values and has mean μ and variance σ^2.

(a) Use Markov's inequality to obtain an upper bound for $P(X \geq \mu + 2\sigma)$. 10

(b) Use Chebyshev's inequality to obtain an upper bound for $P(|X - \mu| \geq 2\sigma)$. 10

For each of cases (a) and (b), obtain the upper bound and the exact probability if $X \sim Exp(\theta)$. 30

4. Let $X_1, X_2, X_3, \ldots$ be a sequence of independent random variables each with $Bernoulli(p)$ distribution (where $0 < p < 1$) and let $\overline{X}_n = \frac{1}{n}\sum_{j=1}^{n} X_j$. State $E[\overline{X}_n]$ and $Var(\overline{X}_n)$.

Use Chebyshev's inequality to obtain an upper bound for $P(|\overline{X}_n - p| \geq 0.1p)$. Hence show that $P(|\overline{X}_n - p| \geq 0.1p) \leq 0.05$ for $n \geq \frac{2000(1-p)}{p}$. 20

It is known that $0.25 < p < 0.75$. Use the Central Limit Theorem to find the value of n (in terms of p) so that $P(|\overline{X}_n - p| \geq 0.1p) \doteq 0.05$. What is the smallest sample size n you would need to choose if you want $P(|\overline{X}_n - p| \geq 0.1p) \leq 0.05$ for all $0.25 < p < 0.75$? 30

(Note that the upper 2.5% point of the $N(0,1)$ distribution is 1.96)