

## Probability 2 - MINIMAL REQUIREMENTS FOR EXAM.

This is a brief list of questions to which you should pay particular attention while preparing for the exam. The word “minimal” in the title of this list explains why some (equally important) questions are not listed below. Apart of these requirements you should be able to solve problems similar to those solved in CW’s.

**Notes 1.** (a) Know probability generating functions (p.g.f.) for standard discrete distributions.  
(b) Know how to find  $P(X = k)$  by differentiating the p.g.f.  
(c) Know how to find  $E[X]$  and  $Var(X)$  by differentiating the p.g.f.  
(d) Know that the p.g.f. of the sum of independent r.v.’s is the product of the individual p.g.f.’s.

**Notes 2.** (a) Know the Theorem of Total Probability and be able to do simple examples.

(b) Be able to prove the Theorem of Total Probability.

(c) Know the gambler’s ruin result:  $r_k \equiv r_k(M, N) = \frac{(\frac{q}{p})^k - (\frac{q}{p})^M}{(\frac{q}{p})^N - (\frac{q}{p})^M}$  if  $p \neq q$  and  $r_k = \frac{k-M}{N-M}$  if  $p = q = \frac{1}{2}$ .

Know similar formulae for  $l_k(M, N)$ .

Be able to use the results in simple examples.

**Notes 3.** (a) Know the Theorem of Total Probability for Expectations. Be able to do simple examples.

(b) Be able to prove the Theorem of Total Probability for Expectations.

(c) In the gambler’s ruin problem, let  $T_k$  be the duration of the game. Set  $E_k = E[T_k]$ . Prove the following

**Theorem.** *The expectations  $E_k$  satisfy the following difference equations:*

$$E_k = 1 + pE_{k+1} + qE_{k-1}, \text{ if } M < k < N; E_M = E_N = 0.$$

**Notes 4.** Branching processes (BP).

(a) Know the definition of the BP  $Y_n$  with generating r.v.  $X$ .

(b) Be able to find the p.g.f. for  $Y_n$  for small  $n$  in simple examples using  $G_{n+1}(t) = G(G_n(t))$

(c) Be able to find  $\theta_n = P(Y_n = 0)$  for small  $n$  in simple examples using  $\theta_{n+1} = G(\theta_n)$ .

(d) Be able to find the probability of eventual extinction  $\theta$  by solving  $G(t) = t$ .

(e) Know the statements of all theorems concerned with BP. Be able to prove theorems 1 and 3.

(f) Read carefully part 6 of Notes 4. Understand how does one find  $E(Y_n)$  and  $Var(Y_n)$  if  $Y_0 = k$   
*Exercise:* find  $E(Y_n)$  if  $Y_0$  is a random variable.

**Notes 5.** Know results on conditional distributions and conditional expectations and be able to use this for simple examples.

In particular know:

$$E[Y] = E[E[Y|X]] \text{ and } \text{Var}(Y) = E[\text{Var}(Y|X)] + \text{Var}(E[Y|X])$$

Know and be able to prove the following result for random sums using these formulae:

If  $S = \sum_{j=1}^N X_j$  where the  $X_j$ 's are i.i.d. with common mean  $a$  and common variance  $\sigma^2$  and  $N$  is a random variable which is independent of the  $X_j$ 's, then:

$$E[S] = aE[N] \text{ and } \text{Var}(S) = \sigma^2 E[N] + a^2 \text{Var}(N)$$

**Notes 6.** Know the definition of the Trinomial Distribution and its simple properties.

**Notes 7.** Continuous random variables.

(a) Define what is a continuous random variable? Define the probability density function (p.d.f.) of a random variable.

(b) How does one compute  $E[g(X)]$  (and in particular  $E(X^k)$ ) in terms of  $f_X(x)$ ?

(c) Define a moment generating function (m.g.f.). Know m.g.f.'s for such distributions as Uniform, Exponential, Gamma, Normal.

(d) Be able to find  $E[X]$  and  $\text{Var}(X)$  by expanding or differentiating the m.g.f.

**Notes 8.** (a) What are jointly continuous random variables?

(b) How do you find the marginal p.d.f.'s in terms of  $f_{X,Y}(x,y)$ ?

(c) Understand all examples discussed in Notes 8.

**Notes 9.** (a) Be able to state all basic definitions, facts, theorems, and understand examples from Notes 9.

(b) Be able to prove Theorem 2 and its following corollary: If  $X$  and  $Y$  are independent,  $Z = X + Y$  then  $M_Z(t) = M_X(t)M_Y(t)$ .

**Notes 10.** (a) Know all basic definitions, facts, theorems from Notes 10.

(b) Know the definition of  $f_{Y|X}(y|x)$  and be able to solve problems similar examples on page 2 of these notes.

**Notes 12.** (a) Know all statements and be able to prove all Lemmas and theorems related to the LLN.

(b) Be able to state the CLT.