# Orthogonally Decomposable Tensors

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Tensors, their Decompositions, and Applications
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# Symmetric Tensors

$T$ is an $\underbrace{n \times \ldots \times n}_{d \text{ times}}$ *symmetric tensor* with elements in $\mathbb{R}$ if

$$T_{i_1 i_2 \ldots i_d} = T_{i_{\sigma_1} i_{\sigma_2} \ldots i_{\sigma_d}}$$

for all permutations $\sigma$ of $\{1, 2, \ldots, d\}$. Notation: $T \in S^d(\mathbb{R}^n)$.

Example $(d = 2)$

$$T = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{22} & \cdots & T_{2n} \\ & & \vdots & \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix}$$

Example $(n = 3, d = 3)$

$$T = \underbrace{\begin{pmatrix} T_{111} & T_{112} & T_{113} \\ T_{112} & T_{122} & T_{123} \\ T_{113} & T_{123} & T_{133} \end{pmatrix}}_{T_{1\cdot\cdot}}, \underbrace{\begin{pmatrix} T_{112} & T_{122} & T_{123} \\ T_{122} & T_{222} & T_{223} \\ T_{123} & T_{223} & T_{233} \end{pmatrix}}_{T_{2\cdot\cdot}}, \underbrace{\begin{pmatrix} T_{113} & T_{123} & T_{133} \\ T_{123} & T_{223} & T_{233} \\ T_{133} & T_{233} & T_{333} \end{pmatrix}}_{T_{3\cdot\cdot}}.$$

# Symmetric Tensors and Polynomials

An equivalent way of representing a symmetric tensor $T \in S^d(\mathbb{R}^n)$ is by a *homogeneous polynomial* $f_T \in \mathbb{R}[x_1, ..., x_n]$ of degree $d$.

## Example ($d = 2$)

In the case of matrices,

$$\begin{aligned} f_T(x_1, ..., x_n) &= x^T T x \\ &= \begin{pmatrix} x_1 & x_2 & \cdots & x_n \end{pmatrix} \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{12} & T_{13} & \cdots & T_{2n} \\ & & \vdots & \\ T_{1n} & T_{2n} & \cdots & T_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \\ &= \sum_{i,j} T_{ij} x_i x_j. \end{aligned}$$

# Symmetric Tensors and Polynomials

For general $T \in S^d(\mathbb{R}^n)$,

$$f_T(x_1, \ldots, x_n) = T \cdot x^d := \sum_{i_1,\ldots,i_d=1}^{n} T_{i_1 \ldots i_d} x_{i_1} \ldots x_{i_d}$$

# Symmetric Tensors and Polynomials

For general $T \in S^d(\mathbb{R}^n)$,

$$f_T(x_1, ..., x_n) = T \cdot x^d := \sum_{i_1,...,i_d=1}^{n} T_{i_1 ... i_d} x_{i_1} ... x_{i_d}$$

$$= \sum_{j_1+\cdots+j_n=d} \binom{d}{j_1, \ldots, j_n} T_{\underbrace{1 \ldots 1}_{j_1} \ldots \underbrace{n \ldots n}_{j_n}} x_1^{j_1} \ldots x_n^{j_n}$$

$$= \sum_{j_1+\cdots+j_n=d} u_{j_1,...,j_n} x_1^{j_1} \ldots x_n^{j_n}.$$

## Example $(n = 3, d = 2)$

For $3 \times 3$ matrices,

$$f_T(x_1, x_2, x_3) = \sum_{i_1,i_2=1}^{3} T_{i_1 i_2} x_{i_1} x_{i_2}$$

$$= \underbrace{T_{11}}_{u_{2,0,0}} x_1^2 + \underbrace{2T_{12}}_{u_{1,1,0}} x_1 x_2 + \underbrace{2T_{13}}_{u_{1,0,1}} x_1 x_3 + \underbrace{T_{22}}_{u_{0,2,0}} x_2^2 + \underbrace{2T_{23}}_{u_{0,1,1}} x_2 x_3 + \underbrace{T_{33}}_{u_{0,0,2}} x_3^2.$$

# Symmetric Tensor Decomposition

A *symmetric decomposition* of a symmetric tensor $T \in S^d(\mathbb{R}^n)$ is

$$T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d}.$$

If $f_T \in \mathbb{R}[x_1, ..., x_n]$ is the corresponding polynomial, then

$$f_T(x_1, ..., x_n) = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d = \sum_{i=1}^{r} \lambda_i (v_{i1}x_1 + v_{i2}x_2 + \cdots + v_{in}x_n)^d.$$

The smallest $r$ for which such a decomposition exists is the *symmetric rank* of $T$.

# Orthogonal Tensor Decomposition

An *orthogonal symmetric decomposition* of a symmetric tensor $T \in S^d(\mathbb{R}^n)$ is a decomposition

$$T = \sum_{i=1}^{r} \lambda_i v_i^{\otimes d} \quad \text{with corresponding} \quad f_T = \sum_{i=1}^{r} \lambda_i (v_i \cdot x)^d$$

such that the vectors $v_1, ..., v_r \in \mathbb{R}^n$ are orthonormal. In particular, $r \leq n$.

## Definition

A tensor $T \in S^d(\mathbb{R}^n)$ is *orthogonally decomposable*, or *odeco*, if it has an orthogonal decomposition.

# Examples

1. All symmetric matrices are odeco: by the spectral theorem

$$T = V^T \Lambda V = \begin{bmatrix} | & \cdots & | \\ v_1 & \cdots & v_n \\ | & \cdots & | \end{bmatrix} \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \begin{bmatrix} - & v_1 & - \\ & \vdots & \\ - & v_n & - \end{bmatrix}$$

$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes 2},$$

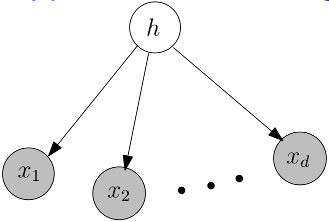
   where $v_1, ..., v_n$ is an orthonormal basis of eigenvectors.

2. The Fermat polynomial: If $v_i = e_i$, for $i = 1, ..., n$, then

$$f_T(x_1, ..., x_n) = x_1^d + x_2^d + \cdots + x_n^d,$$

$$T = e_1^{\otimes d} + e_2^{\otimes d} + \cdots + e_n^{\otimes d}.$$

# An Application: Exchangeable Single Topic Models




Pick a topic $h \in \{1, 2, ..., k\}$ with distribution $(w_1, ..., w_k) \in \Delta_{k-1}$. Given $h = j$, $x_1, ..., x_d$ are *i.i.d* random variables taking values in $\{1, 2, ..., n\}$ with distribution $\mu_j = (\mu_{j1}, ..., \mu_{jn}) \in \Delta_{n-1}$.

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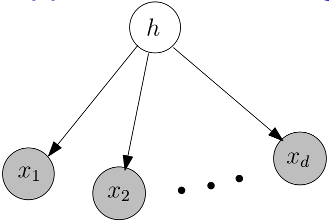


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Then, the joint distribution of $x_1, ..., x_d$ is an $\underbrace{n \times n \times \cdots \times n}_{d \text{ times}}$ symmetric tensor $T \in S^d(\mathbb{R}^n)$ whose entries sum to 1. Moreover,

$$T = \sum_{j=1}^{k} \mathbb{P}(h = j)\mathbb{P}(x_1 | h = j) \otimes \cdots \otimes \mathbb{P}(x_d | h = j) = \sum_{j=1}^{k} w_j \mu_j^{\otimes d}.$$

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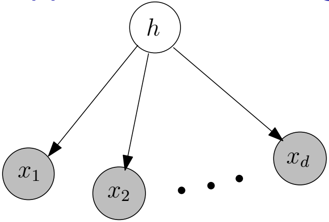


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Given $T$, to recover the parameters $w, \mu$, use a transformation $T \mapsto T_{od}$ and decompose $T_{od}$ [Anandkumar et al.].

# Eigenvectors of Tensors

Consider a symmetric tensor $T \in S^d(\mathbb{R}^n)$.

Example ($d = 2$)

$T$ is an $n \times n$ matrix and $w \in \mathbb{C}^n$ is an eigenvector if

$$Tw = \begin{pmatrix} \vdots \\ \sum_{j=1}^n T_{i,j} w_j \\ \vdots \end{pmatrix} = \lambda w.$$

Example ($d = 3$)

$T$ is an $n \times n \times n$ tensor and $w \in \mathbb{C}^n$ is an eigenvector if

$$Tw^2 := \begin{pmatrix} \vdots \\ \sum_{j,k=1}^n T_{i,j,k} w_j w_k \\ \vdots \end{pmatrix} = \lambda w.$$

# Eigenvectors of Symmetric Tensors

## Definition

- Given a symmetric tensor $T \in S^d(\mathbb{R}^n)$, an *eigenvector* of $T$ with *eigenvalue* $\lambda$ is a vector $w \in \mathbb{C}^n$ such that

$$Tw^{d-1} := \begin{pmatrix} \vdots \\ \sum_{i_2,\dots,i_d=1}^{n} T_{i,i_2,\dots,i_d} w_{i_2} \dots w_{i_d} \\ \vdots \end{pmatrix} = \lambda w.$$

  Two eigenvector-eigenvalue pairs $(w, \lambda)$ and $(w', \lambda')$ are equivalent if there exists $t \in \mathbb{C} \setminus \{0\}$ such that $t^{d-2}\lambda = \lambda'$ and $tw = w'$.

- For the corresponding $f_T \in \mathbb{R}[x_1, \dots, x_n]$, $w \in \mathbb{C}^n$ is an *eigenvector* with *eigenvalue* $\lambda$ if

$$\nabla f_T(w) = d\lambda w.$$

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$$\nabla f_T(w) = d\lambda w.$$

Therefore, the eigenvectors of $f$ are given by the vanishing of the

$$2 \times 2 \text{ minors of the matrix } [\nabla f_T(x)|x].$$

# Eigenvectors of Symmetric Tensors

## Example

Let $d = 2$ and $T$ be an $n \times n$ symmetric matrix. Then

$$f_T(x_1, \ldots, x_n) = \sum_{i,j} T_{ij} x_i x_j.$$

Thus,

$$\nabla f_T(x_1, \ldots, x_n) = \begin{pmatrix} 2(\sum_{i=1}^n T_{1i} x_i) \\ 2(\sum_{i=1}^n T_{2i} x_i) \\ \vdots \\ 2(\sum_{i=1}^n T_{ni} x_i) \end{pmatrix} = 2Tx.$$

So, $x$ is an eigenvector with eigenvalue $\lambda$ if and only if $\nabla f_T(x) = 2\lambda x$.

# Eigenvectors of Symmetric Tensors

## Example

Let

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3} \text{ and } f_T(x,y,z) = x^3 + y^3 + z^3.$$

Then, $(x,y,z)^T$ is an eigenvector of $f_T$ if and only if the $2 \times 2$ minors of the matrix $\begin{bmatrix} & x \\ \nabla f & y \\ & z \end{bmatrix} = \begin{bmatrix} 3x^2 & x \\ 3y^2 & y \\ 3z^2 & z \end{bmatrix}$ vanish. Therefore,

$$x^2y - xy^2 = x^2z - xz^2 = y^2z - yz^2 = 0.$$

This is equivalent to

$$xy(x-y) = xz(x-z) = yz(y-z) = 0.$$

The solutions are (up to scaling):

$$\{(1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)\}.$$

# Eigenvectors of Odeco Tensors

If $T = \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}$ is an odeco tensor, i.e. $v_1, ..., v_n$ are orthonormal, then the vectors $v_k$, $k = 1, ..., n$ are eigenvectors of $T$ with corresponding eigenvalues $\lambda_k$, $k = 1, ..., n$:

$$T v_k^{d-1} = \sum_{i=1}^{n} \lambda_i v_i (v_k \cdot v_i)^{d-1} = \lambda_k v_k.$$

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- Is there an easy way of finding these vectors, i.e. finding the orthogonal decomposition of an odeco tensor?
- Are these all of the eigenvectors of an odeco tensor?

# Robust Eigenvectors

## Definition

A unit vector $u \in \mathbb{R}^n$ is a *robust eigenvector* of a tensor $T \in S^d(\mathbb{R}^n)$ if there exists $\epsilon > 0$ such that for all $\theta \in \mathcal{B}_\epsilon(u) = \{u' : ||u - u'|| < \epsilon\}$, repeated iteration of the map

$$\theta \mapsto \frac{T\theta^{d-1}}{||T\theta^{d-1}||}, \tag{1}$$

starting from $\theta$ converges to $u$.

## Theorem (Anandkumar et al.)

*Let $T$ have an orthogonal decomposition $T = \sum_{i=1}^{k} \lambda_i v_i^{\otimes d}$ with $v_1, \ldots, v_k$ orthonormal, and assume that $\lambda_1, \ldots, \lambda_k > 0$.*

1. *The set of $\theta \in \mathbb{R}^n$ which do not converge to some $v_i$ under repeated iteration of (1) has measure 0.*
2. *The set of robust eigenvectors of $T$ is equal to $\{v_1, v_2, ..., v_k\}$.*

# The Tensor Power Method

## Algorithm

Input: An odeco tensor $T$.
Output: An orthogonal representation of $T$.

Repeat

Find $v_i \leftarrow$ power method output starting from a random $u \in \mathbb{R}^n$.
Recover $\lambda_i = T \cdot v_i^d$.
$T \leftarrow T - \lambda_i v_i^{\otimes d}$.

Return $v_1, ..., v_k$ and $\lambda_1, ..., \lambda_k$.

The tensor power method consists of repeated iteration of the map

$$u \mapsto \frac{Tu^{d-1}}{||Tu^{d-1}||},$$

or equivalently,

$$u \mapsto \frac{\nabla f(u)}{||\nabla f(u)||}.$$

# The Number of Eigenvectors of a Tensor

<u>Recall</u>: Given a tensor $T \in S^d(\mathbb{C}^n)$ with corresponding polynomial $f_T$, the eigenvectors $x \in \mathbb{C}^n$ are the solutions to the equations given by the $2 \times 2$ minors of the matrix

$$\left[\nabla f_T(x) | x\right].$$

## Theorem (Sturmfels and Cartwright)

*If a tensor $T \in S^d(\mathbb{C}^n)$ has finitely many eigenvectors, then their number is $\frac{(d-1)^n - 1}{d-2}$.*

# Eigenvectors of Odeco Tensors

Odeco tensors are nice because we can characterize all of their eigenvectors.

## Theorem

*Let* $T \in S^d(\mathbb{C}^n)$ *be an odeco tensor with* $d \geq 3$ *and* $T = \sum_{i=1}^n \lambda_i v_i^{\otimes d}$*. Then,* $T$ *has* $\frac{(d-1)^n - 1}{d-2}$ *eigenvectors. Explicitly, they are*

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \lambda_{\sigma(1)}^{-\frac{1}{d-2}} v_{\sigma(1)} + \eta_2 \lambda_{\sigma(2)}^{-\frac{1}{d-2}} v_{\sigma(2)} + \cdots + \eta_k \lambda_{\sigma(k)}^{-\frac{1}{d-2}} v_{\sigma(k)},$$

*where* $k = 1, ..., n$, $\eta_2, ..., \eta_k$ *are* $(d-2)$*-nd roots of unity and* $\sigma$ *is a permutation on* $\{1, \ldots, n\}$.

# Eigenvectors of Odeco Tensors

### Example ($d = 3, n = 3$)

Let

$$T = e_1^{\otimes 3} + e_2^{\otimes 3} + e_3^{\otimes 3}.$$

Then, $V = I$, the identity matrix and the eigenvectors of $T$ are:

$k = 1$ $(1:0:0)^T, (0:1:0)^T, (0:0:1)^T$

$k = 2$ $(1:1:0)^T, (1:0:1)^T, (0:1:1)^T$

$k = 3$ $(1:1:1)^T.$

# The Set of Odeco Tensors

- Parametric representation:
  The set of orthogonally decomposable tensors can be parametrized by $\mathbb{R}^n \times O_n(\mathbb{R})$:
  $$\lambda, V \mapsto \sum_{i=1}^{n} \lambda_i v_i^{\otimes d}.$$
- Implicit representation:
  The set of orthogonally decomposable tensors can also be represented as the solutions to a set of equations.

## Definition

The *odeco variety* is the set of all odeco tensors in $S^d(\mathbb{R}^n)$.

Goal: find equations defining this variety.

# The Odeco Variety

Let $T \in S^d(\mathbb{R}^n)$. For $q = 1, \ldots, d$, consider the tensor $T *_q T \in S^2(S^{d-1}(\mathbb{R}^n))$.

# The Odeco Variety

Let $T \in S^d(\mathbb{R}^n)$. For $q = 1, \ldots, d$, consider the tensor $T *_q T \in S^2(S^{d-1}(\mathbb{R}^n))$. Let $\mathcal{F}$ be the set of equations defining by the condition

$$T *_q T \in S^{2(d-1)}(\mathbb{R}^n).$$

## Theorem (Boralevi,Draisma,Horobeţ,R.)

*The odeco variety is equal to zero set of $\mathcal{F}$ for every $n$.*

## Conjecture

The ideal defined by $\mathcal{F}$ is prime.

# Odeco Tensors as Algebras

Let $T \in S^3(\mathbb{R}^n)$ be a symmetric tensor of order 3. Let $V_T = \mathbb{R}^n$ be equipped with a positive definite inner product $(\cdot|\cdot)$. Then, $V$ has the structure of an algebra via the product

$$\begin{aligned} V_T \times V_T &\to V_T \\ (x, y) &\mapsto x \star y := T(x, y, \cdot). \end{aligned}$$

The inner product is compatible with the product in the sense that

$$(x \star y|z) = T(x, y, z) = (z \star x|y).$$

And the product is commutative

$$x \star y = y \star x.$$

## Theorem (Boralevi,Draisma,Horobeţ,R.)

*The tensor $T$ is odeco if and only if $(V, \star)$ is associative, i.e.*

$$x \star (y \star z) = (x \star y) \star z.$$

# Nonsymmetric Tensor Decomposition

Let $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n = (\mathbb{R}^n)^{\otimes d}$. A *decomposition* of $T$ is an expression of the form

$$T = \sum_{i=1}^{r} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots .$$

A tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ is *orthogonally decomposable*, or *odeco*, if we can decompose it as

$$T = \sum_{i=1}^{n} \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots ,$$

so that $a_1, \ldots, a_n \in \mathbb{R}^n$ are orthonormal, $b_1, \ldots, b_n \in \mathbb{R}^n$ are orthonormal, $c_1, \ldots, c_n \in \mathbb{R}^n$ are orthonormal, etc.

# Nonsymmetric Odeco Tensors

## Example

1. If $T \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a matrix, then $T$ has singular value decomposition

$$T = U \Sigma V^T = \sum_{i=1}^{k} \sigma_i u_i v_i^T = \sum_{i=1}^{k} \sigma_i u_i \otimes v_i,$$

where $u_1, \ldots, u_k$ are orthonormal and $v_1, \ldots, v_k$ are orthonormal.

2. The tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ given by

$$T = \sum_{i=1}^{n} \lambda_i e_i \otimes e_i \otimes \cdots \otimes e_i$$

is odeco.

# Singular Vector Tuples

## Example

Given a matrix $T \in \mathbb{R}^n \otimes \mathbb{R}^n$, $(u, v)$ is a *singular vector tuple* of $T$ if there exists $\lambda$ such that

$$Tu = \begin{bmatrix} \vdots \\ \sum_j T_{ij} u_j \\ \vdots \end{bmatrix} = \lambda v \qquad \text{and} \qquad T^T v = \begin{bmatrix} \vdots \\ \sum_i T_{ij} v_i \\ \vdots \end{bmatrix} = \lambda u.$$

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## Definition

Given a tensor $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ a *singular vector tuple* is a $d$-tuple $(x_1, \cdots, x_d) \in \mathbb{C}^n \times \cdots \times \mathbb{C}^n$ such that for every $1 \leq k \leq d$,

$$T(x_1, \ldots, x_{k-1}, \cdot, x_{k+1}, \ldots, x_d) = \lambda x_k,$$

for some $\lambda \in \mathbb{C}$.

# Example

1. If $T \in \mathbb{R}^n \otimes \mathbb{R}^n$ is a generic matrix with singular value decomposition

$$T = U\Sigma V^T = \sum_{i=1}^{n} \sigma_i u_i v_i^T,$$

$(u_1, v_1), ..., (u_n, v_n)$ are all of the singular vector pairs of $T$.

2. Let $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$ be odeco with

$$T = \sum_{i=1}^{n} \lambda_i a_i \otimes b_i \otimes c_i.$$

Then, $(a_1, b_1, c_1), \ldots, (a_n, b_n, c_n)$ are singular vector triples, but there are many more additional ones.

# Tensor Power Method for Nonsymmetric Odeco Tensors

Start with odeco $T = \sum \lambda_i a_i \otimes b_i \otimes c_i$.

While $T \neq 0$ repeat

Choose $x^{(0)}, y^{(0)}, z^{(0)} \in \mathbb{R}^n$.

For $i$ from 1 to $N$ repeat

$$x^{(i+1)} = T(\cdot, y^{(i)}, z^{(i)})$$

$$y^{(i+1)} = T(x^{(i)}, \cdot, z^{(i)})$$

$$z^{(i+1)} = T(x^{(i)}, y^{(i)}, \cdot).$$

End for

Find

$$\lambda = T(x^{(N)}, y^{(N)}, z^{(N)}).$$

Set

$$T = T - \lambda x^{(N)} \otimes y^{(N)} \otimes z^{(N)}.$$

End while

## Lemma

With probability 1 the tensor power method converges to one of the $(a_i, b_i, c_i)$. And each of them has a positive probability of happening.

# Singular Vectors of Odeco Tensors

## Theorem

*Let $T \in \mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$ be odeco with decomposition $T = \sum_{i=1}^n \lambda_i a_i \otimes b_i \otimes c_i \otimes \cdots$. Let $A = \begin{pmatrix} a_1 & | & a_2 & | & \cdots | a_n \end{pmatrix}$, $B = \begin{pmatrix} b_1 & | & b_2 & | & \cdots | b_n \end{pmatrix}$, etc., so that $A, B, C, \ldots$ are orthogonal matrices. Then, the singular vector tuples of $T$ are given as follows:*

Type I

$$\left( A \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \chi_{12}\eta_2\lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \chi_{1k}\eta_k\lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, B \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \chi_{22}\eta_2\lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \chi_{2k}\eta_k\lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, C \begin{bmatrix} \lambda_1^{-\frac{1}{d-2}} \\ \chi_{32}\eta_2\lambda_2^{-\frac{1}{d-2}} \\ \vdots \\ \chi_{3k}\eta_k\lambda_k^{-\frac{1}{d-2}} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots \right),$$

*where $1 \leq k \leq n$, $\chi_{ij}$ is a 2-nd root of unity, $\eta_i$ is a $(d-2)$-nd root of unity, up to permutation.*

Type II

$$(Ax_1, Bx_2, \ldots, Cx_3, \ldots),$$

*where the matrix $X = (x_{ij})_{ij}$ has at least two zeros in each column and no row is identical to $0$.*

# The Set of Odeco Tensors

## Definition

The *odeco variety* is the Zariski closure of the set of all odeco tensors in $\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n$.

For $q = 1, \ldots, d$ consider $T *_q T \in S^2(\mathbb{R}^{n_1} \otimes \cdots \otimes \mathbb{R}^{n_{q-1}} \otimes \mathbb{R}^{n_{q+1}} \otimes \cdots \otimes \mathbb{R}^{n_d})$.

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Let $\mathcal{F}$ be the deal defined by the condition that for every $q = 1, \ldots, d$

$$T *_q T \in S^2(\mathbb{R}^{n_1}) \otimes \cdots \otimes S^2(\mathbb{R}^{n_{q-1}}) \otimes S^2(\mathbb{R}^{n_{q+1}}) \otimes \cdots \otimes S^2(\mathbb{R}^{n_d}).$$

## Theorem (Boralevi, Draisma, Horobeţ, R.)

*The set of orthogonally decomposable tensors equals $\mathcal{V}(\mathcal{F})$.*

## Conjecture

The ideal $\mathcal{F}$ is prime.

# Decomposing Tensors into Frames

A general tensor $T \in S^d(\mathbb{R}^n)$ has rank $\lfloor \frac{1}{n}\binom{n+d-1}{d} \rfloor$.
An odeco tensor $T = \sum_{i=1}^n \lambda_i v_i^{\otimes d}$ has rank $n$.
Question: How to enlarge the set of odeco tensors to contain tensors of higher ranks?

Idea: Let $V := \left(\mathbf{v}_1, \cdots, \mathbf{v}_r\right) \in (\mathbb{R}^n)^r$ be a *finite unit norm tight frame*, i.e.

$$VV^T = \frac{r}{n} I_n \quad \text{and} \quad ||\mathbf{v}_j||^2 = 1, j = 1, ..., r.$$

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$$VV^T = \frac{r}{n} I_n \quad \text{and} \quad ||\mathbf{v}_j||^2 = 1, j = 1, ..., r.$$

A tensor $T \in S^d(\mathbb{R}^n)$ is *frame decomposable* (or *fradeco*) if it can be written as

$$T = \sum_{i=1}^r \lambda_i \mathbf{v}_i^{\otimes d},$$

where $(\mathbf{v}_1, ..., \mathbf{v}_r)$ form a finite unit norm tight frame.

# Finite Unit Norm Tight Frames

## Examples

- The Mercedes Benz Frame $V = \begin{pmatrix} 0 & \frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} \\ 1 & -\frac{1}{2} & -\frac{1}{2} \end{pmatrix}$.

- $V = \frac{1}{3\sqrt{3}} \begin{pmatrix} -5 & 1 & 1 & 3 \\ 1 & -5 & 1 & 3 \\ 1 & 1 & -5 & 3 \end{pmatrix}$.

- $V = \begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 1 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix}$.

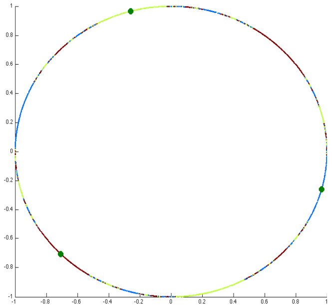
# The tensor power method

## Conjecture

Let $r = n + 1$ and $T = \sum_{j=1}^{n+1} \lambda_j \mathbf{v}_j^{\otimes d}$ with $\lambda_1, ..., \lambda_{n+1} > 0$. Then, $\mathbf{v}_1, ..., v_{n+1}$ are the robust eigenvectors of $T$, so they are found by the tensor power method.

## Example (The Mercedes Benz Frame)

Let $T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes 5} + \begin{pmatrix} \frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}^{\otimes 5} + \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ -\frac{1}{2} \end{pmatrix}^{\otimes 5}$. The dynamics of the power method looks like this



# The tensor power method

## Example

Let $n = 2, r = 4, d = 5$ and consider the tensor

$$T = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix}^{\otimes 4} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}^{\otimes 4} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}^{\otimes 4} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}^{\otimes 4},$$

where $\alpha > 6$. The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is an eigenvector, but none of the other eigenvectors are real. Therefore, the frame decomposition of $T$ cannot be recovered from its eigenvectors.

# References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