If two were three, what would Hex be?

Alex Fink*

Hex is an abstract strategy board game with appealingly simple rules, invented by Piet Hein in 1942 and independently by John Nash in 1947, and first disseminated by Martin Gardner in *Mathematical Games* [4].

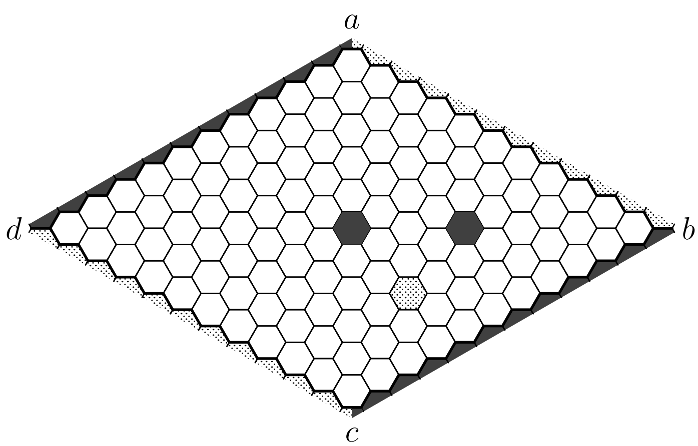

Figure 1: A recently begun game of Hex.

Hex is played by two players, Black and White, on an equilateral rhombic board composed of hexagonal cells, often 11 to a side, though Nash recommended 14. A possible early-game position is shown in Figure 1; here and henceforth we represent White territory with speckled cells. A small fringe around the outside of the board is coloured as in the figure from the beginning, associating two sides of the rhombus with each player. When the game begins, the cells of the rhombus are empty. At each turn, the player to move must fill a currently unfilled cell with their own colour. So the position depicted in the figure could have arisen after three plays, assuming Black moved first.

The game finishes when every cell of the board is coloured. (Formally, at least; if the winner becomes clear before the board fills you're welcome to stop the game early.) When the board is entirely coloured we consider the common boundaries, or *borders*, of black and white regions. Points on the borders lie primarily along the edges of cells. Some cell vertices will be border points as

*finka@math.berkeley.edu

well, and for each border vertex exactly two of the edges issuing from it will be borders. So the borders form a collection of disjoint nonforking paths. These are loops except for those which run into one of the endpoints a, b, c, d in Figure 1.

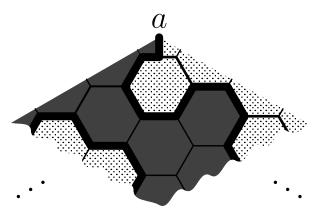

Figure 2: Borders in a corner of a finished game of Hex.

These four endpoints are joined in two pairs by the borders. Borders can't cross—nor can it change which colour is to your left side as you walk along a border from end to end—so the pairing that joins a with c and b with d is impossible. That leaves either a with b and c with d, in which case we declare Black to be the winner, or a with d and c with b, in which event we give the win to White. In other words, *a player wins in Hex if the borders follow the board edges not of their own colour.* One of these two pairings is always established, so this proves that every game has a unique winner.

"But wait", you may be saying if you've seen Hex before, "isn't the objective to connect your two sides with cells of your colour?" Indeed it is; this is a restatement of the same goal. If Black (say) is successful in building a path of black stones connecting the two black sides of the board, cutting off a and b from c and d, they've thereby prevented White from running a border across this path, and assured themself victory. Conversely in any game won by Black, the Black cells verging on the border that joins a to b connect the two black sides. That is: *a player wins in Hex if they separate the border endpoints two and two according to their own pairing by a path of stones of their own colour*; and such a separating path connects the two sides of the board of their colour.

This interpretation in terms of forming connections is of course the original and widely known one, on account of which abstract games in the genre represented by Hex are known as *connection games.* The border interpretation which we take here as primary is first suggested in David Gale's treatment [3].

In fact many connection games can have their winning condition stated in terms of borders, though in the majority of cases the resulting rule is rather forced. The less artificial end of the gamut is represented for instance by Mark Steere's Atoll [6], whose winning condition can also be stated in terms of establishing groups of pairwise connections among endpoints, and by Rombo, which we take up below.

Some months ago, during teatime discussions in the Berkeley math department lounge, the idea arose of generalising Hex to three players, or three dimensions, I forget which first; the feeling soon descended on us that the two properties would go together nicely. But these initial explorations stayed within the connection game framework, and we never hit upon a satisfying ruleset, one which preserved the property that a unique winner should exist. Some time later it occurred to me to try to generalise the rules based on borders instead, and in this vein I came to invent the game Trunc, which is my subject here[1]. My investigations of Trunc are still preliminary; I haven't gotten my head around the game enough to play reasonably, or to speak to any points of strategy. This writeup therefore centres on the process of invention.

Take any three-dimensional board divided into cells, and consider a finished three-colour game. The analogue of our borders in two dimensions, which we call borders still, will now be points which are on the common boundary of all three colours. The borders lie principally along cell edges, and will form a collection of nonforking paths so long as one is careful not to allow more than two to meet at a vertex. In 2D, Hex arranged this by choosing a hexagonal tiling, with only three edges emanating from each vertex. Four edges would have been too many, for all four could be borders (this is why Hex played on the square tiling so readily ends without either player making a connection). In 3D the least number of convex polyhedral cells that can meet at a vertex is four, achieved among uniform tilings of three-space only by the truncated octahedral tiling of Figure 3. Again, this minimum number of cells at a vertex, i.e. four, is as many as we can allow; any way to fit five cells around a vertex allows four borders to meet there. So to ensure that the borders form paths, Trunc shall be played on a board of truncated octahedra. This is of course the source of the name *Trunc* — just like *Hex*, it's a truncation of the name of the cell shape.

Figure 3: Left: the truncated octahedron. Right: a piece of its space-filling tessellation; several vertices common to four cells are visible.

The truncated icosahedral tiling can also be seen as the *Voronoi tessellation* of the *body-centered cubic lattice*. The body-centered cubic lattice is the set of all points in $\mathbb{Z}^3$ whose coordinates all have the same parity. Its points are the cell centres. The cells themselves partition space around these so that each point belongs to the cell whose centre it's closest to: that is, each cell is the set of all points closer to the centre of a cube than to any corner.

[1]I claim a connection to our theme *eight*, that is *two to the three*: I'm investigating what becomes of Hex when one changes each *two to a three*!

In general the tiling by n-dimensional (regular) *permutohedra*, of which the hexagonal and truncated octahedral tilings are the two- and three-dimensional cases, has only $n+1$ cells meeting at each vertex, and so could serve as a tiling on which to base a n-dimensional game. Indeed, in his consideration [3] of the Brouwer theorem in dimensions $n > 2$, Gale prefigures this entire class of board shapes (including the Trunc board), specifying their duals as the graphs on $\mathbb{Z}^n$ whose edges are all the parallel translates of nonzero vectors in $\{0, 1\}^n$. Gale even goes on to invoke n-player n-dimensional games of Hex, but for $n > 2$ his games permit multiple winners, and so for our purposes are unsatisfying.

Before moving on, we touch on a rather different way a three-dimensional connection game might have a border-based winning condition, found in Cameron Browne's Rombo [2]. In order for borders in three dimensions to be pathlike, they must consist of the points on the boundary of a three-colouring of space — but not all three of these colours must belong to a player! So suppose we have only two players, and take unfilled cells as our third "colour". Then, if the filled cells form a simply connected clump, all the borders will be visible from the outside and lie on the surface of this clump, where they'll be just the dividing lines between black and white.

Rombo is a two-player game, played on an infinite tiling of rhombic dodecahedra. The game begins with a small cluster of four filled cells, two of each colour, with exactly one border loop. The players alternately add cells to this cluster until a second border loop is created. At this point, one of the players' colours is on the inside of both loops, as viewed from unfilled space, and the other is on the outside; the latter player is declared the winner. The rhombic dodecahedral board does allow four border segments to meet at a point, so Trunc had to reject it; but since Rombo isn't concerned with using the borders to pair points up, it can simply shrug this off. Nonetheless, when four border segments do meet at a point, one might defensibly choose to count the resulting configuration of borders as either one loop or two, and Browne recognises both variants.

Trunc takes a cube for the shape of the whole board — actually, not quite, but this'll do for two paragraphs. The faces of a cube can be three-coloured symmetrically, in the colours of the three players White, Gray, and Black, so that all three meet around each vertex, providing endpoints for the borders. (We couldn't have done this three-colouring with a tetrahedron, say.) By happy coincidence it's possible to find good cubical subregions of the tiling: this is easy to see in terms of the body-centered cubic lattice, which already has the symmetries of the cube around any point. A cubical subregion is exemplified in Figure 4. Again, to give the border endpoints prominence, a layer of cells surrounding the board is coloured from the outset, with the cells verging on an edge or a vertex split into two or three coloured regions, respectively.

Our cube has four border endpoints of each handedness: around the unprimed corners the colors are arranged cyclically in one sense, around the primed corners in the other. Each border must join two endpoints of opposite handed-

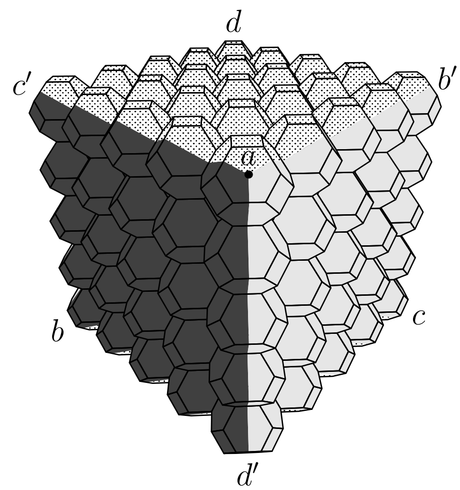

Figure 4: A small cubical board, from the outside. Corner a' is hidden.

ness. If we colour in all the cells arbitrarily, the borders might join the primed to the unprimed corners in any of the $4! = 24$ conceivable ways. But there's no satisfactory way to assign a winner to each of these. No player deserves to win more than any other when the borders (or even a single border) join any corner to the diametrically opposite one, since there are symmetries preserving this configuration but replacing any colour outside the board by any other.

Trunc resolves this difficulty by moving to a *quotient space*: that is, we glue (or formally, *identify*) parts of the board together (think of gluing together edges of a rolled-up rectangle to form a cylinder, say). The invocation of a quotient space has some precedents among existing connection games, among which are Mark Thompson's Gaia [1, pp. 110–112] and Bill Taylor's Projex, both played on projective planes.

In the case of Trunc we identify all pairs of opposite points in the cube: this means sticking whole three-dimensional volumes of space together, point to corresponding point, so it might be a bit tricky to carry out as a gluing unless you have a fourth dimension handy. This identification gives the Trunc board a singular point at the center of the cube around which the space angle is 2π steradians, half its usual value. The continuous version of the board is no longer a smooth manifold but an *orbifold*, i.e. locally (in our case even globally) the quotient of Euclidean space by a finite group action, here the action of a cyclic group of order 2 by reflection about the cube's center. The boundary of the board is the *hemi-cube*, an *abstract polytope* of 4 vertices, 6 edges, and 3 faces. Abstract polytopes are objects with the combinatorial structure of polytopes, but which fail to be embeddable in Euclidean space like polytopes are: the hemi-cube for instance is homeomorphic as a surface to the projective plane. The hemi-cube is portrayed again more cleanly in Figure 5.

Figure 5: Left: the hemicube as double cover of the cube. Right: the hemicube inside the projective plane; opposite points on the boundary of the disk are identified.

If you're not a fan of quotient spaces—or even if you are—feel free to interpret the game as played on an unidentified board with a rule that any move must colour in two diametrically opposite regions, unless it takes the very middle region.

The identification unifies each unprimed border endpoint with its primed partner at the opposite end of a body diagonal. We use the unprimed letter for the identified point. In a finished game the endpoints are still joined in pairs: the singular point at the centre causes no problems, since it's in the middle of a cell and so no borders pass through it. This means a can't be joined to a', since these two are now the same point; thus we've gotten rid of all the cases that were giving us problems defining a winning condition, where the borders followed body diagonals. On the other hand any of the four identified border endpoints can be joined to any other, since the board has become non-orientable. This gives three possible pairings of border endpoints: a—b and c—d; a—c and b—d; or a—d and b—c. In Trunc each of the three players has one of these as winning condition: *a player wins in Trunc if the borders follow the board edges not incident to their own colour.* For instance Black's objective is to join the border endpoints so as to pair a with b and c with d.

The goal of Trunc can be restated in the connection game spirit, just as for Hex, by focusing not on the borders but on the structures of filled cells keeping them apart.

In three dimensions, to contain a one-dimensional border and prevent it escaping, we must consider a two-dimensional surface. Hopefully it's intuitively clear what a surface of cells is; if not, you may wish to think of it as the cells cut by some surface in real space that meets no vertices of the board, or as a collection of cells such that any pair of adjacent cells (not on the boundary) forms part of two triangles.

Say that a surface of cells is *impermeable* if it contains no triangles in which all three colours appear. An impermeable surface is impermeable to borders, by the very definition of borders. So if Black, say, can create an impermeable surface cutting off a and b from c and d, the borders can only join the endpoints in these pairs and Black has won. The converse holds too: if Black has won, then there's some impermeable surface cutting off a and b from c and d. It's easy to choose one that doesn't intersect any border; the borders that are closed loops can be freely placed on either side. That is: *a player wins in Trunc if they separate the border endpoints two and two, into the pairs they want to connect with borders, by an impermeable surface*[2].

One possible winning impermeable surface for Black, seen darkly through a glass which smoothes out the discreteness, meets the boundary of the board in the dotted loop in the right side of Figure 5. The surface itself, distorted, could look something like Figure 6.

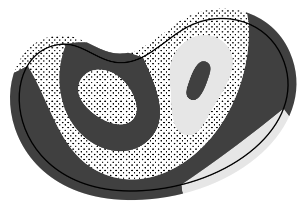

Figure 6: Possible colouration of an impermeable surface, with a fringe coloured as the boundary of the board.

The definition of an impermeable surface conspicuously lacks any condition on the colour of the cells it contains that would associate it with the player it wins for. Indeed, a majority of the cells comprising a winning impermeable surface S for Black may be nonblack. But there is necessarily at least a path of black stones connecting the two (pre-identification) black faces within S, which we might call the surface's *trunk*. This can be established by an argument about two-colour borders internal to S, each of which must verge on the same two colours along its length, and one of which must run from black face to black face with the trunk alongside it. Complications arise since there can exist a multiplicity of termination points of such borders, one at every place the boundary of S crosses an edge of the board. We leave the details for the reader to fill in.

The behaviour of borders in three dimensions suggests a family of two-dimensional two-player games, of which Hex is one member, which are of interest in their own right.

Think of the gray cells adjacent to the gray side of the board as an outgrowth of the cells outside the board, such that all the cells adjacent to the outgrowth are either white or black. The borders cleave to this outgrowth, and so they behave as they would in the two-dimensional two-player game played on the surface of cells containing the adjoining white and black cells. If the outgrowth contains no connection between the two gray sides that existed before identification, i.e.

[2] With this winning condition in mind Richard Guy suggests the alternate name *Waldorf*: one aims to ensure that two of the vertices are *walled off* from the other two.

its boundary is an orientable surface, then just as in regular two-dimensional Hex it's impossible that a be joined to c, by dint of orientation. Thus if Gray has won, the gray outgrowth must contain this connection (this is another approach to seeing the existence of the trunk). On the other hand if one of the other players, say Black, has won the game of Trunc, then in particular they have won the game played against White on the gray outgrowth, and symmetrically the game against Gray on the white one.

Note that these two-player games don't actually arise within Trunc as such, since the third player can interact with them, changing the effective board shape, while the game is still ongoing. From the perspective of Trunc, which we momentarily set aside, these two-player games can at best be taken as alternate simplificatory ways of looking at the goal.

If the gray outgrowth does have orientable boundary, it may *a fortiori* even lack handles, in which case the game on its surface is topologically equivalent to Hex, differing only in the tiling, and so is little different in flavour. But if there is a handle, the character of the game changes more essentially. Not only do we lose the obvious connection interpretation of Hex (both White and Black may establish connections), but we lose even the fundamental property that *making a move in your colour never hurts you.* This property is the key ingredient in the non-constructive proof that Hex is a first player win, and hence the proof fails on handled surfaces.

Figure 7 exemplifies the simplest counterexample, with only a single handle. I've drawn the board with spatial symmetries exchanging black and white to underscore the symmetry of the position. Black has taken the entire top of the handle, which runs left to right in the foreground, and White the entire valley under the handle, which runs top to bottom and is mostly obscured. In particular both players already have made end-to-end connections. There remains just one cell left to fill, on the handle's underside. Check that whichever of White and Black fills this cell loses.

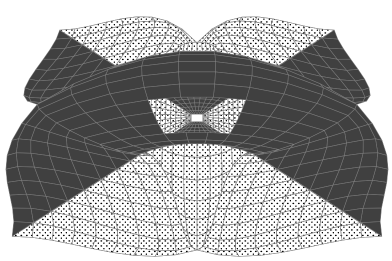

Figure 7: A position of the game on the one-handled surface, to be lost by the next mover. A hole has been cut in the handle to reveal the one unfilled cell.

More generally, positions of games on handled surfaces are a generalisation of Misère Hex, which has been discussed by a number of previous authors, among them by Jeffrey Lagarias and Daniel Sleator in the collection from the first Gathering for Gardner [5]. They determine which of the two players has the win for each board size, invoking a monotonicity property which assigns a preference ranking to the possible states of each cell and says that changing a single cell by a more preferred state can never hurt the player it's for. ("Making a move never hurts you" is another example of a monotonicity property.)

Given a board with one large handle, the regular Hex board can be embedded along the handle's underside, with the rest of the board solidly coloured in two pieces on the model of Figure 7. Then connecting a to b and c to d on the embedded board, which is Black's usual winning condition, will win the larger game for White, and vice versa. Examples of this nature establish that no monotonicity properties can hold globally in games on handled surfaces, and so we don't expect any analysis like Lagarias and Sleator's to extend to this larger class.

If you find this loss of monotonicity undesirable, you might compensate by introducing a variant rule which allows any player to make a move of any colour. This gives each player the same set of options from each position; of course, the underlying game remains *partisan*, i.e. sensitive to who's Black and who's White, since the evaluation of finished positions isn't colorblind. In Hex the extra options this variant grants never help the player to move, and so don't affect the strategy at all, which suggest that either the ruleset with the variant or the one without it could be the "correct" generalisation of the rules of Hex. In two-player games on handled surfaces, the variant eliminates all $\mathcal{P}$-positions, or Zugzwangs, such as the one of Figure 7, so that if a player can win from a position with *some* player to move, then they can in fact make a winning move themself. Check that our construction to give Misère Hex as a position of a game with one handle in fact now reduces to normal Hex.

The position of Figure 7 embeds in a Trunc endgame, showing that in Trunc too it's possible to make a move which hurts you. So one might apply our variant rule to Trunc as well. This again would eliminate the class of positions where the player to move can't win, regardless of who that is. But these are not the only class of potentially distasteful position. Like many three-player games, *kingmaker* positions arise in which one player can't win but may decide which of the other two will, and the variant rule does nothing to address these. For example, Figure 8 depicts the two unfilled cells of a Trunc game two moves from completion. The cells surrounding them are coloured as suggested by the face colouration in the figure, and the border endpoints touching the unfilled cells are connected to the board corners as labelled. If Gray is to move from this position followed by Black, then, with the variant rule or without, Gray is in a kingmaker position, as you can check.

This writeup has left a number of pragmatic points bearing on the gameplay unaddressed — What's a suitable board size? Is a pie rule still necessary, to correct for any first-player advantage which may exist, and if so what's the correct implementation for three players? How is one even supposed to go about building or otherwise manipulating the three-dimensional board? — but my hope is that interested players will come upon reasonable answers to these. I myself

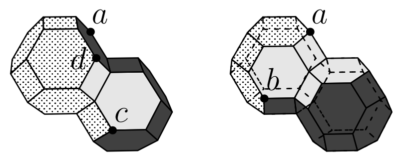

Figure 8: Two unfilled cells which constitute a kingmaker position for Gray, front (left) and inside (right) views.

would be keenly interested to hear of any discoveries, strategic or mathematical, about Trunc or these games on handled surfaces.

I am grateful to Cameron Browne for informing me of precedents among connection games of some of the ideas here.

References

[1] Cameron Browne, *Connection Games: Variations on a Theme*, A. K. Peters, 2005.

[2] Cameron Browne, *Rombo*, `http://www.cameronius.com/games/rombo/`.

[3] David Gale, The Game of Hex and the Brouwer Fixed-Point Theorem, *Amer. Math. Monthly* **86** (1979), pp. 818–827.

[4] Martin Gardner, *The Scientific American Book of Mathematical Puzzles and Diversions*, Simon and Schuster, New York, 1959, pp. 73–83.

[5] Jeffrey Lagarias and Daniel Sleator, "Who Wins Misère Hex". In *The Mathemagician and Pied Puzzler: A Collection in Tribute to Martin Gardner*, ed. E. Berlekamp and T. Rodgers, A. K. Peters, 1999, pp. 237–240.

[6] Mark Steere, *Atoll — A Newly Discovered Principle*, `http://marksteere.blogspot.com/2008/01/atoll-newly-discovered-principle.html`.