Isometries between JB*-triples

Cho-Ho Chu¹, Michael Mackey²

¹ School of Mathematical Sciences, Queen Mary, University of London, London UK
E1 4NS (e-mail: c.chu@qmul.ac.uk)
² Department of Mathematics, University College Dublin, Belfield, Dublin 4, Ireland
(e-mail: michael.mackey@ucd.ie)

Received: 19 July 2004; in final form: 27 January 2005 / Published online: 10 August 2005 – © Springer-Verlag 2005

Abstract. Let Z and W be JB*-triples and let T be a linear isometry from Z into W. For any z ∈ Z with ∥z∥ < 1, we show that

\[ T\{z, z, z\} = \{T(z), T(z), T(z)\} \]

if the Möbius transform induced by T(z) preserves the unit ball of T(Z). We show further that T is, locally, a triple homomorphism via a tripotent: for any z ∈ Z, there is a tripotent u in W** such that

\[ \{u, T\{a, b, c\}, u\} = \{u, \{T(a), T(b), T(c)\}, u\} \]

for all a, b, c in the smallest subtriple Z_z of Z containing z, and also, \{u, T(·), u\}: Z_z → W** is an isometry.

Mathematics Subject Classification (2000): 46L70, 32M15, 46G20, 17C65, 46L05

1. Introduction

Jordan algebraic structures play an important role in the geometry of infinite dimensional Banach manifolds. Indeed, as shown by Kaup [14], every bounded symmetric domain gives rise to a Jordan triple product {·, ·, ·} on its tangent space and a surjective linear map T between these spaces is an isometry if, and only if, it preserves the Jordan triple product:

\[ T\{a, b, c\} = \{T(a), T(b), T(c)\} \].

These tangent spaces form an important class of complex Banach spaces, called JB*-triples. One can therefore study the geometry of symmetric domains via the
algebraic structures of JB*-triples. We remark that Jordan methods were first introduced by Koecher [15] into the theory of finite dimensional bounded symmetric domains and they were also discussed in detail in [16].

Although a Jordan triple monomorphism is necessarily an isometry, a non-surjective linear isometry between two JB*-triples need not preserve the Jordan triple product. It is natural to ask to what extent can a non-surjective linear isometry preserve the Jordan triple product. The object of this paper is to address this question. We note that, by polarization, a linear map \( T : Z \rightarrow W \) between JB*-triples preserves the Jordan triple product if, and only if,

\[
T\{a,a,a\} = \{T(a), T(a), T(a)\} \quad (a \in Z).
\]

To answer the above question, our first task is to understand what makes a surjective linear isometry preserve the Jordan triple product. Upon a closer study of the geometry behind the proof of this fact in [14, Proposition 5.5], we found that the condition needed is a certain invariant property of the Möbius transformation. In Section 3 we discuss this in detail and show that, given a linear isometry \( T : Z \rightarrow W \) between JB*-triples, not necessarily surjective, one has

\[
T\{a,a,a\} = \{T(a), T(a), T(a)\}
\]

for \( \|a\| < 1 \) if the Möbius transformation \( g_{T(a)} \) induced by \( T(a) \) preserves the open unit ball of the image \( T(Z) \). In Section 4, we show that, although a non-surjective linear isometry \( T : Z \rightarrow W \) between JB*-triples need not be a triple homomorphism, it is, nevertheless, locally a triple homomorphism, that is, for any \( a \in Z \), there is a tripotent \( u \in W^{**} \) such that \( \|\{u, T(z), u\}\| = \|z\| \) and

\[
\{u, T[z, z, z], u\} = \{u, \{T(z), T(z), T(z)\}, u\}
\]

for every \( z \) in the JB*-triple generated by \( a \). The tripotent \( u \) above depends on the given element \( a \in Z \), but if \( Z \) admits a character, then one can find a tripotent \( v \in W^{**} \) such that \( \{v, T(\cdot), v\} \neq 0 \) and

\[
\{v, T[z, z, z], v\} = \{v, \{T(z), T(z), T(z)\}, v\}
\]

for all \( z \in Z \). Without any condition on \( Z \), such a tripotent \( v \) may not exist. Finally in Section 5, we prove more specialized results in the setting of JB*-algebras. In particular, we show that, if \( T : Z \rightarrow W \) is a linear isometry from a JB*-triple \( Z \) into a JB*-algebra \((W, \circ)\), then there is a largest projection \( p \in W^{**} \) such that, for all \( a \in Z \),

\[
T\{a,a,a\} \circ p = \{T(a), T(a), T(a)\} \circ p
\]

and \( p \) operator commutes with \( T(a) \circ T(a)^* \).

The results in this paper generalize those in [7] for C*-algebras. We begin in the next section with some basic definitions and results concerning JB*-triples.
2. JB*-triples

Throughout this paper, an isometry \( T : Z \to W \) between Banach spaces is not assumed to be surjective and we often write \( T a \) for the image \( T(a) \) for convenience. We first recall that a JB*-triple \( Z \) is a complex Banach space equipped with a continuous Jordan triple product \( \{ \cdot, \cdot, \cdot \} : Z^3 \to Z \) which is symmetric and linear in the outer variables, and conjugate linear in the middle variable such that for \( a, b, c, x, y \in Z \), we have

(i) \( \{ a, b, \{ c, x, y \} \} = \{ \{ a, b, c \}, x, y \} - \{ c, \{ b, a, x \}, y \} + \{ c, x, \{ a, b, y \} \} \);

(ii) the map \( z \in Z \mapsto \{ a, a, z \} \in Z \) is hermitian with nonnegative spectrum;

(iii) \( \| \{ a, a, a \} \| = \| a \|^3 \).

For later reference, we define two fundamental linear operators on a JB*-triple \( Z \). For \( x, y \in Z \), the box operator \( x \square y : Z \to Z \) and the Bergman operator \( B(x, y) : Z \to Z \) are defined by

\[
(x \square y)(z) = \{ x, y, z \},
\]

\[
B(x, y)(z) = z - 2\{ x, y, z \} + \{ x, \{ y, z, y \}, x \}.
\]

Every C*-algebra \( A \) is a JB*-triple with the following Jordan triple product

\[
\{ a, b, c \} = \frac{1}{2} (ab^*c + cb^*a) \quad (a, b, c \in A).
\]

A closed subspace of a JB*-triple is called a subtriple if it is closed with respect to the triple product. A linear map \( T : Z \to W \) between JB*-triples is called a triple homomorphism if it preserves the triple product in which case, the kernel \( J \) of \( T \) is a triple ideal of \( Z \), that is, \( \{ Z, Z, J \} + \{ Z, J, Z \} \subseteq J \) and the range \( T(Z) \) is a subtriple of \( W \). We refer to [3, 6, 18–20] for expositions as well as recent surveys of JB*-triples and symmetric Banach manifolds. In the sequel, we write \( a^{(3)} = \{ a, a, a \} \) and use frequently the polarization formula

\[
\{ a, b, c \} = \frac{1}{8} \sum_{a^i b^j = 1} a^i b^j (a + ab + bc)^{(3)}.
\]

An element \( u \) in a JB*-triple is called a tripotent if \( u^{(3)} = u \). If a JB*-triple \( Z \) has a predual (which is necessarily unique), then it is called a JBW*-triple in which case, \( Z \) has an abundance of tripotents. Each tripotent \( u \in Z \) induces a splitting of \( Z \), \( Z = Z_0 \oplus Z_1 \oplus Z_2 \), known as the Peirce decomposition, into a direct sum of the 0, 1 and 2-eigenspaces of the operator \( 2u \square u \). The Peirce projections \( P_i(u) : Z \to Z_i \) onto the eigenspaces \( Z_i \), for \( i = 0, 1, 2 \), are given in terms of the triple product,

\[
P_0(u)(z) = B(u, u)(z)
\]

\[
P_1(u)(z) = 2\{ u, u, z \} - \{ u, \{ u, z, u \}, u \}
\]

\[
P_2(u)(z) = \{ u, \{ u, z, u \}, u \}.
\]

These projections are contractive. Each eigenspace \( Z_i \) is a subtriple of \( Z \). Indeed we have \( \{ Z_i, Z_j, Z_k \} \subset Z_{i+j+k} \) for \( i, j, k \in \{ 0, 1, 2 \} \) where \( Z_r := \{ 0 \} \) for \( r \not\in \{ 0, 1, 2 \} \).
In particular, $Z_2 = P_2(u)(Z)$ is a JB*-algebra with identity $u$, and with respect to the following non-associative product and involution:

$$x \circ y = \{x, u, y\}, \quad x^* = \{u, x, u\}.$$  

We note that a JB*-algebra, that is, a Jordan Banach algebra $(A, \circ)$ equipped with an isometric involution $\ast$ satisfying $\|x \circ y\| \leq \|x\| \|y\|$ and $\|\{x, x, x\}\| = \|x\|^3$, is also a JB*-triple with the Jordan triple product

$$\{a, b, c\} = (a \circ b)^* \circ c - (a \circ c) \circ b^* + (b^* \circ c) \circ a.$$  

For example, a C*-algebra is a JB*-algebra with the Jordan product $a \circ b = \frac{1}{2}(ab + ba)$.

A JB*-algebra is called a JC*-algebra if it can be embedded as a norm-closed subspace of a C*-algebra, closed with respect to the involution and the above Jordan product. A JB*-algebra having a (necessarily unique) predual is called a JBW*-algebra, it is called a JW*-algebra if it is also a JC*-algebra. We refer to [10] for a detailed exposition of Jordan Banach algebras including JB*-algebras and JBW*-algebras.

Each tripotent $u$ in a JBW*-triple $Z$ has a support face $F(u)$ in the predual $Z_*$ of $Z$, given by

$$F(u) = \{\varphi \in Z_* : \|\varphi\| = 1 = \varphi(u)\}$$

which is a norm-exposed face of the closed unit ball $Z_{a1}$ of $Z_*$. One can introduce a partial ordering $\leq$ to the set $T(Z)$ of tripotents in a JBW*-triple $Z$. For any two tripotents $u$ and $v$ in $Z$, one defines $u \leq v$ if $v - u$ is orthogonal to $u$ which means that

$$\{u, v - u, x\} = 0$$

for all $x \in Z$. With this partial ordering, it has been shown in [8] that given a family of tripotents $\{u_\alpha\}_{\alpha \in Q}$ in $Z$, either the lattice supremum $\bigvee_{\alpha \in Q} u_\alpha$ exists in $T(Z)$, or $Z_{a1} = \bigvee_{\alpha \in Q} F(u_\alpha)$, that is, the smallest norm-exposed face of $Z_{a1}$ containing the union $\bigcup_{\alpha \in Q} F(u_\alpha)$ is $Z_{a1}$ itself. By [5], $Z$ embeds as a subtriple of a JBW*-algebra $A$ such that the predual $Z_*$ is a 1-complemented subspace of the predual $A_*$ of $A$, where we recall that a closed subspace of a Banach space $E$ is called 1-complemented if it is the range of a contractive projection on $E$. In particular, faces of $Z_{a1}$ are faces of the closed unit ball $A_{a1}$ and, every face $F$ of $A_{a1}$ is either disjoint from $Z_{a1}$ or the intersection $F \cap Z_{a1}$ is a face of $Z_{a1}$. It follows that, if $\{u_\alpha\}_{\alpha \in Q}$ is a family of tripotents in $T(Z)$ such that $Z_{a1} = \bigvee_{\alpha} F(u_\alpha)$, then we also have $A_{a1} = \bigvee_{\alpha} F(u_\alpha)$, where $\bigvee$ denotes the supremum in $A_{a1}$, for otherwise, $F = \bigvee_{\alpha} F(u_\alpha)$ is a proper norm-exposed face of $A_{a1}$ and the intersection $F \cap Z_{a1}$ is a norm-exposed face of $Z_{a1}$ containing $\bigcup_{\alpha} F(u_\alpha)$, giving $F \cap Z_{a1} = Z_{a1}$ which is impossible since $0 \not\in F$. 


By [8, p.322], every element \( z \) in a JBW*-triple \( Z \) admits a support tripotent \( u_z \in T(Z) \) satisfying
\[
z = \{u_z, z, u_z\} = \{u_z, u_z, z\}.
\]

3. Isometries and Möbius transformation

In this section, we reveal the role of Möbius transformations in the preservation of Jordan structures by a linear isometry. We first introduce the relevant geometric and holomorphic aspects of JB*-triples. A map \( g : D \rightarrow U \) between open sets in complex Banach spaces \( Z \) and \( W \), respectively, is called holomorphic if the Fréchet derivative \( g'(a) : Z \rightarrow W \) exists for every \( a \in D \), where \( g'(a) \) is a linear map satisfying
\[
\lim_{t \to 0} \frac{\|g(a + t) - g(a) - g'(a)(t)\|}{\|t\|} = 0.
\]
A holomorphic map \( g : D \rightarrow U \) is called biholomorphic if it is bijective and the inverse \( g^{-1} \) is also holomorphic. The open unit ball of a Banach space \( Z \) will be denoted by \( Z_0 \). Let \( Aut Z_0 \) be the automorphism group of \( Z_0 \), consisting of all biholomorphic maps from \( Z_0 \) onto itself. Upmeier [20] has shown that \( Aut Z_0 \) is a real Banach-Lie group and by a deep result of Kaup [14], a complex Banach space \( Z \) is a JB*-triple if, and only if, \( Aut Z_0 \) acts transitively on \( Z_0 \), in which case, the Jordan triple product is constructed via the Lie algebra of \( Aut Z_0 \). For a JB*-triple \( Z \), the basic elements in \( Aut Z_0 \) are the Möbius transformations. Given \( a \in Z_0 \), we define the Möbius transformation of \( Z_0 \), induced by \( a \), to be the biholomorphic map \( g_a : Z_0 \rightarrow Z_0 \) given by
\[
g_a(z) = a + B(a, a)^{1/2}(I + z \Box a)^{-1}(z)
\]
where \( I \) is the identity operator. We have \( g_a(0) = a, g_a^{-1} = g_{-a} \) and, the Fréchet derivatives \( g'_a(0) = B(a, a)^{1/2} \) and \( g'^{-1}_a(a) = B(a, a)^{-1/2} \) (cf. [14]). If \( Z \) is a C*-algebra, we have the following formula for the Möbius transformation which was due to Potapov [17] and Harris [11]:
\[
g_a(z) = (1 - aa^*)^{-1/2}(a + z)(1 + a^*z)^{-1}(1 - a^*a)^{1/2}.
\]

**Lemma 1.** Let \( T : Z \rightarrow W \) be a linear isometry between JB*-triples \( Z \) and \( W \). Let \( a \in Z_0 \) and let \( \psi \in Aut T(Z)_0 \) be such that \( \psi(T(a)) = 0 \). Then
\[
\psi(0) = -\psi'(T(a)) \left( T(B(a, a)^{1/2}(a)) \right).
\]

**Proof.** Let \( h = \psi T g_a : Z_0 \rightarrow T(Z)_0 \). Then \( h \) is biholomorphic and \( h(0) = 0 \). Hence \( h \) is linear by Cartan’s uniqueness theorem and on \( Z_0 \),
\[
h = h'(0) = (\psi T g_a)'(0) = (\psi T)'(g_a(0))g'_a(0) = (\psi T)'(a)B(a, a)^{1/2}.
\]
Evaluating \( h \) at \(-a\), we get the formula.
We note that $T g_{-a} T^{-1}$ is an automorphism of $T(Z)_0$ and maps $T(a)$ to 0. For a C*-algebra, we have $B(a, a)(1/2)(a) = (1 - a a^*)^{1/2} a (1 - a^* a)^{1/2} = a - a a^* a$ since $(1 - a a^*)^{1/2} a = a (1 - a^* a)^{1/2}$. Therefore we have $B(a, a)(1/2)(a) = a - [a, a, a]$ in a JB*-triple by considering the subtriple generated by $a$ which is linearly isometric to an abelian C*-algebra. Since $B(a, a)(1/2)(a) = a - a a^* a$, it follows that, if $D = -D$ is a subset of the open unit ball of a JB*-triple, invariant under $g_a$, then it is also invariant under $g_{-a}$ and $g_a(D) = D$.

By refining Kaup's result in [14, Proposition 5.5] (see also [11]), we now show how the Möbius transformation and surjectivity effect the preservation of the triple product by a linear isometry.

**Proposition 1.** Let $T : Z \to W$ be a linear isometry between JB*-triples $Z$ and $W$. Let $a \in Z_0$ and let $g_{T(a)} \in Aut W_0$ be the Möbius transformation induced by $T(a)$. If $g_{T(a)}(T(Z)_0) \subset T(Z)_0$, then we have

$$T(a, a, a) = (T(a), T(a), T(a)).$$

In particular, if $T$ is surjective, then $T$ is a triple isomorphism.

**Proof.** Let $\psi$ be the restriction to $T(Z)_0$ of the Möbius transformation $g_{-T(a)} \in Aut Z_0$. Then $\psi \in Aut T(Z)_0$, $\psi(T(a)) = 0$ and the derivative $\psi'(T(a)) : T(Z) \to T(Z)$ is the restriction of the derivative $g'_{T(a)}(T(a)) : W \to W$ which is equal to $B(T(a), T(a))^{-1/2}$. By Lemma 1, we have

$$-T(a) = \psi'(0) = -\psi'(T(a))B(T(a, a)^{1/2}(a)) = -B(T(a), T(a))^{-1/2}T(a - a^{(3)}).$$

It follows that $T(a) - T(a)^{(3)} = B(T(a), T(a))^{1/2}(T(a)) = T(a - a^{(3)})$ which gives $T(a)^{(3)} = T(a^{(3)})$.

Finally, if $T$ is surjective then $T(Z)_0 = W_0$ is invariant under $g_{T(a)}$ for all $a \in A_0$. Hence $T$ preserves the triple product.

**Remark 1.** The above result subsumes Kadison's seminal result for surjective isometries between C*-algebras. It has also been discussed in [4] in the setting of JB*-algebras. We note from [7] that, for a fixed $a$, the condition $T(a)^{(3)} = (T(a))^{(3)}$ alone does not imply $T(a^{(n)}) = (T(a))^{(n)}$ for any odd integer $n > 3$.

The following corollary is immediate.

**Corollary 1.** Let $T : Z \to W$ be a linear isometry between JB*-triples. Then $T(Z)$ is a subtriple of $W$ if, and only if, $T(Z)_0$ is invariant under the Möbius transformation $g_{T(a)}$ for all $\|a\| < 1$.

**Example 1.** Let $M_n$ be the JB*-triple of $n \times n$ complex matrices. Let $T : \mathbb{C} \to M_2$ be defined by

$$T(a) = \begin{pmatrix} 0 & a \\ \frac{a}{2} & 0 \end{pmatrix}.$$
Then $T$ is a linear isometry and $T(\mathbb{C})$ is not a subtriple of $M_2$. Also $T(1)$ is not unitary and $T(\mathbb{C})$ contains no nontrivial positive element. For $a \neq 0$, we have $T(a^{(3)}) \neq (Ta)^{(3)}$ and in fact, for $0 < |a| < 1$,

$$g_{Ta}(Tx) = \left(\begin{array}{cc}
0 & \frac{2a(1)}{1 + 2a(0)} \\
\frac{a(1)}{1 + 2a(0)} & 0
\end{array}\right)$$

which is outside $T(\mathbb{C})$.

**Example 2.** A Hilbert space $H$ is a JB*-triple with Jordan triple product

$$[x, y, z] = \frac{1}{2} (\langle x, y \rangle z + \langle z, y \rangle x)$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $H$. Hence, given a linear isometry $T : H \rightarrow K$ between Hilbert spaces, the range $T(H)$ is a subtriple of $K$ and $T$ is a triple isomorphism onto $T(H)$.

Given $a \in H_0$, the Möbius transformation $g_a : H_0 \rightarrow H_0$ is given by

$$g_a(x) = a + E_a(x) + \sqrt{1 - \|a\|^2} (I - E_a)(x)$$

where $E_a$ is the projection from $H$ onto the subspace $\mathbb{C}a$. Given a linear isometry $T$ on $H$, we have $\langle Tx, Ta \rangle = \langle x, a \rangle$ and $E_{Ta}(Tx) = E_a(x)Ta$. It follows that

$$g_{Ta}(Tx) = T(g_a(x))$$

and indeed, $T(H)_0$ is invariant under $g_{Ta}$ for all $\|a\| < 1$.

**Example 3.** Let $C(\Omega)$ and $C(\Omega \cup \{\beta\})$ be the C*-algebras of continuous functions on the closed unit disc $\Omega \subset \mathbb{C}$ and $\Omega \cup \{\beta\}$ respectively, where $\beta \in \mathbb{C} \setminus \Omega$. Define $T : C(\Omega) \rightarrow C(\Omega \cup \{\beta\})$ by

$$(Ta)(x) = \begin{cases} 
a(x) & \text{if } x \in \Omega \\
\frac{1}{2}(a(1) + a(0)) & \text{if } x = \beta.
\end{cases}$$

Then $T$ is a linear isometry and $T(C(\Omega)) = \{h \in C(\Omega \cup \{\beta\}) : 2h(\beta) = h(1) + h(0)\}$ which is not a subtriple of $C(\Omega \cup \{\beta\})$. It is easy to see that $T$ is not a triple isomorphism onto its range, but for $a \in C(\Omega)_0$ with $a(1) = a(0) = 0$, we have $g_{Ta}(T(C(\Omega))_0) \subset T(C(\Omega))_0$. Indeed, if $h \in T(C(\Omega))_0$, then

$$g_{Ta}(h)(\beta) = \frac{Ta + h}{1 + Ta h}(\beta) = \frac{Ta(\beta) + h(\beta)}{1 + Ta(\beta)h(\beta)} = \frac{2(a(1) + a(0) + h(1) + h(0))}{4 + 2(a(1) + a(0))(h(1) + h(0))} = \frac{1}{2} (h(1) + h(0))$$

which gives $g_{Ta}(h) \in T(C(\Omega))_0$. It is clear that $T(a^{(3)}) = T(a)^{(3)}$. 

Isometries between JB*-triples

621
4. Isometries and Jordan triple product

Our goal in this section is to show that a non-surjective linear isometry \( T : Z \rightarrow W \) between JB*-triples preserves, at least locally, the Jordan triple product, via a tripo-tent. Since the JB*-subtriple generated by an element \( z \in Z \) is Jordan isomorphic to the JB*-triple \( C_0(X) \) of complex continuous functions on a locally compact Hausdorff space \( X \), vanishing at infinity, it suffices to study the case in which \( Z = C_0(X) \).

We recall that for any functional \( \varphi \) in the predual of a JBW*-triple \( W \), there is a unique tripotent \( u_\varphi \in W^* \) called the support tripotent of \( \varphi \), such that \( \varphi = \varphi \circ P_2(u_\varphi) \) and \( \varphi | P_2(u_\varphi)(W) \) is a faithful normal positive functional on the JBW*-algebra \( P_2(u_\varphi)(W) \) [9, Proposition 2]. The JBW*-algebra \( P_2(u_\varphi)(W) \) becomes an inner product space with respect to the inner product
\[
\langle a, b \rangle = \varphi \{ a, b, u_\varphi \}.
\]
Moreover \( \varphi \) is an extreme point of the closed unit ball of the predual if, and only if, \( u_\varphi \) is a minimal tripotent, that is, \( [u_\varphi, W, u_\varphi] = \mathbb{C} u_\varphi \). We denote by \( \partial E \) the set of extreme points of the closed unit ball of a Banach space \( E \).

As usual, we embed and regard a JB*-triple \( Z \) as a subtriple of its second dual \( Z^{**} \) which is a JBW*-triple. The following theorem generalizes the results in [7, 12].

**Theorem 1.** Let \( W \) be a JB*-triple and let \( T : C_0(X) \rightarrow W \) be a linear isometry. Then either \( T \) is a triple homomorphism or there is a tripotent \( u \in W^{**} \) such that
\[
\{ u, T(f^3), u \} = \{ u, T(f)^3, u \}
\]
for all \( f \in C_0(X) \) and
\[
\{ u, T(\cdot), u \} : C_0(X) \rightarrow W^{**}
\]
is an isometry.

**Proof.** Let \( E = T(C_0(X)) \). Then the dual map \( T^* : E^* \rightarrow C_0(X)^* \) of \( T : C_0(X) \rightarrow E \) is a surjective linear isometry. We also denote by \( T^* \) the dual map of \( T : C_0(X) \rightarrow W \) since no confusion is likely. Let
\[
Q = \{ \varphi \in \partial W^* : \varphi | E \in \partial E^* \}.
\]
Then \( Q \) is non-empty since each extreme point \( \psi \in \partial E^* \) extends to an extreme point \( \varphi \in \partial W^* \).

Let \( \varphi \in Q \) with \( \psi = \varphi | E \in \partial E^* \). Then \( T^* \varphi = T^* \psi \) is an extreme point of the closed unit ball of \( C_0(X)^* \) and hence there exists \( x_\varphi \in X \) such that \( T^* \psi = \alpha x_\varphi \) with \( |\alpha| = 1 \). Let \( u_\varphi \in W^{**} \) be the support tripotent of \( \varphi \).

Since \( u_\varphi \) is a minimal tripotent and \( \varphi[u_\varphi, \cdot, u_\varphi] = \varphi \circ P_2(u_\varphi)(\cdot) = \bar{\varphi}(\cdot) \), where the bar ‘-’ denotes complex conjugation, we have
\[
\{ u_\varphi, b, u_\varphi \} = \overline{\varphi(b)} u_\varphi \quad (b \in W^{**}).
\]
From $\varphi \circ T(f) = (T^* \varphi)(f) = (T^* \psi)(f) = \alpha f(x_\varphi)$, we obtain, in $W^{**}$,
\[
\{u_\varphi, T(f), u_\varphi\} = \alpha f(x_\varphi) u_\varphi \quad (f \in C_0(X))
\]
and $\{u_\varphi, T(\cdot), u_\varphi\}$ is a triple homomorphism. In particular,
\[
\alpha f(x_\varphi) u_\varphi = \{u_\varphi, T(f), u_\varphi\} = \{u_\varphi, \{Tf, P_2(u_\varphi)(Tf), Tf\}, u_\varphi\} = \alpha f(x_\varphi) \{u_\varphi, \{Tf, u_\varphi, Tf\}, u_\varphi\}
\]
and hence $\varphi[u_\varphi, \{Tf, u_\varphi, Tf\}, u_\varphi] = (\alpha f(x_\varphi))^2$ or
\[
\varphi[Tf, u_\varphi, Tf] = (\alpha f(x_\varphi))^2. \tag{1}
\]
We prove that
\[
\{u_\varphi, T(f^{(3)}), u_\varphi\} = \{u_\varphi, (Tf)^{(3)}, u_\varphi\} \quad (f \in C_0(X)).
\]
It suffices to show that
\[
\varphi[u_\varphi, (Tf)^{(3)}, u_\varphi] = \alpha f^{(3)}(x_\varphi).
\]
We first show that
\[
\{u_\varphi, u_\varphi, Th\} = u_\varphi
\]
for $h \in C_0(X)$ satisfying $\|h\| = 1$ and $h(x_\varphi) = \bar{a}$. We have, by the Schwarz inequality [2, Proposition 1.2],
\[
1 = |\varphi(Th)|^2 = |\varphi[u_\varphi, Th, u_\varphi]|^2
\leq \varphi[u_\varphi, u_\varphi, u_\varphi] \varphi[Th, Th, u_\varphi] \leq \|Th\|^2 = \|h\|^2 = 1
\]
giving $\varphi[Th, Th, u_\varphi] = 1$. Let
\[
N_\varphi = \{b \in W^{**} : \varphi[b, b, u_\varphi] = 0\}.
\]
Then we have
\[
N_\varphi = P_0(u_\varphi)(W^{**}) \tag{2}
\]
by [2, p.516]. We show $Th - u_\varphi \in N_\varphi$. Indeed, we have
\[
\varphi[Th - u_\varphi, Th - u_\varphi] = \varphi[Th - u_\varphi, Th, u_\varphi] - \varphi[u_\varphi, Th, u_\varphi] + \varphi[u_\varphi, u_\varphi, u_\varphi] - \varphi[Th, u_\varphi, u_\varphi] = 0
\]
where $\varphi[Th, u_\varphi, u_\varphi] = \varphi[u_\varphi, Th, u_\varphi] = 1$. Hence, by (2), we have $\{u_\varphi, u_\varphi, Th - u_\varphi\} = 0$ and $\{u_\varphi, u_\varphi, Th\} = u_\varphi$.

We next show that $\varphi[Tg, Tg, u_\varphi] = 0$ whenever $g \in C_0(X)$ satisfies $g(x_\varphi) = 0$. We may assume, by Urysohn’s lemma, that $g$ vanishes on a neighbourhood of $x_\varphi$, in which case, we can choose $k \in C_0(X)$ such that $\|k\| = 1$, $k(x_\varphi) = \alpha$ and $kg = 0$. Then $\|k + g\| = 1$ and $(k + g)(x_\varphi) = \alpha$. Therefore, by the above, we
have $T(k + g) + N\varphi = u\varphi + N\varphi =Tk + N\varphi$ which yields $Tg \in N\varphi$, that is, $\varphi(Tg, Tg, u\varphi) = 0$

Now let $f \in C_0(X)$ with $\|f\| = 1$. Pick $h \in C_0(X)$ with $\|h\| = 1$ and $h(x\varphi) = \bar{\alpha}$. Then $(f - \alpha f(x\varphi)h)(x\varphi) = 0$ and therefore we have $Tf - \alpha f(x\varphi)Th \in N\varphi$

and by (2) again,

$$[u\varphi, u\varphi, Tf - \alpha f(x\varphi)Th] = 0$$

giving

$$\{u\varphi, u\varphi, Tf\} = \alpha f(x\varphi)
\{u\varphi, u\varphi, Th\} = \alpha f(x\varphi)u\varphi.$$ 

Moreover, we have

$$\alpha f(x\varphi)[u\varphi, Tf, u\varphi] = [u\varphi, Tf, [u\varphi, u\varphi, Tf]]$$

$$= \{(u\varphi, Tf, u\varphi), [u\varphi, [Tf, u\varphi, u\varphi], Tf] + [u\varphi, u\varphi, [u\varphi, Tf, Tf]]\}$$

$$= \alpha f(x\varphi)[u\varphi, u\varphi, Tf] - \alpha f(x\varphi)[u\varphi, u\varphi, Tf] + [u\varphi, u\varphi, [u\varphi, Tf, Tf]]$$

$$= [u\varphi, u\varphi, [u\varphi, Tf, Tf]]$$

and hence $\varphi[u\varphi, Tf, Tf] = \varphi([u\varphi, u\varphi, [u\varphi, Tf, Tf]]) = \alpha f(x\varphi)\varphi[u\varphi, Tf, u\varphi]$. Therefore we have

$$\varphi[u\varphi, (Tf)^{(3)}, u\varphi] = \alpha f^{(3)}(x\varphi) = \varphi[u\varphi, (T(f)^{(3)}, u\varphi).$$

By the remarks in Section 2, we have two cases:

(i) the lattice supremum $u = \bigvee_{\varphi \in Q} u\varphi$ is a tripotent in $W^{**}$;

(ii) $W^*_1 = \bigvee_{\varphi \in Q} F(u\varphi) = \bigvee_{\varphi \in Q} \{\varphi\}.$

Case 1. The tripotent $u = \bigvee_{\varphi \in Q} u\varphi$ has support face

$$F(u) = \{\psi \in W^* : \|\psi\| = \psi(u) = 1\}$$

which is the normal state space of the atomic JBW*-algebra $P_2(u)(W^{**})$. Let $\rho$ be an extreme point of $F(u)$ with support tripotent $u\varphi$ which is a minimal projection in $P_2(u)(W^{**})$. If we select from $\{u\varphi\}_{\varphi \in Q}$ a maximal subfamily $\{u\varphi\}_{\varphi \in Q'}$ with mutually orthogonal central supports $\{c(u\varphi)\}_{\varphi \in Q'}$, then $u = \sum_{\varphi \in Q'} c(u\varphi)$ where each $P_2(c(u\varphi))(W^{**})$ is a type I JBW*-factor. It follows from [10, Lemma 5.3.2] that
Isometries between JB*-triples 625

Let $S : C_0(X) \to W^{**}$ be the isometry defined by

$$S(f) = [s, Tf, s]^* \quad (f \in C_0(X))$$

where $*$ is the involution in $P_2(u)(W^{**})$. By the above argument, we have

$$\varphi(S(f^3)) = \varphi((Sf)^3).$$

As $\varphi$ is a state of $P_2(u)(W^{**})$, it follows that

$$\rho(T(f^3)) = \varphi([s, Tf, s]^*) = \rho((Tf)^3).$$

Since $\rho \in F(u)$ was arbitrary, we obtain

$$\{u, T(f^3), u\} = \{u, (Tf)^3, u\}.$$

Finally, for any $f \in C_0(X)$, pick $x \in X$ with $\|f\| = |f(x)|$. Let $\psi \in \partial E^*$ with $T^*\psi = \delta_x$, and let $\varphi \in \partial W^*$ be an extension of $\psi$. Then $\varphi \in Q$ and $T^*\varphi = \delta_x$. Hence

$$\|Tf\| \geq \|\{u, Tf, u\}\| \geq \|\{u_{\varphi}, \{u_{\varphi}, \{u, Tf, u\}, \varphi\}\}\|$$

$$= \|\{u_{\varphi}, Tf, \varphi\}\|$$

$$= \|f(x)u_{\varphi}\| = |f(x)| = \|f\|$$

which gives $\|\{u, Tf, u\}\| = \|f\|$.

**Case 2.** Let $W^{**}$ be embedded as a subtriple of a JBW*-algebra $B$ such that $W^*$ is 1-complemented in the predual $B_\ast$. As remarked in Section 2, we have $B_{\ast 1} = \bigvee_{\varphi \in Q} \varphi$.

It follows that there is a subfamily $\{\varphi\}_{\varphi \in Q'}$ such that the atomic part $B_{\varphi}$ of $B$ is a direct sum

$$B_{\varphi} = \bigoplus_{\varphi \in Q'} B(u_{\varphi})$$

where $B(u_{\varphi})$ is the weak*-closed ideal in $B$ generated by $u_{\varphi}$ and is a type I JBW*-factor. Given an extreme point $\rho \in \partial W^*$, it is also an extreme point of $B_{\ast 1}$ and its support tripotent $u_{\rho}$ is in some $B(u_{\varphi})$. As before, $u_{\rho}$ is equivalent to $u_{\varphi}$ via a symmetry in $B$ and it follows that

$$\rho(T(f^3)) = \rho((Tf)^3).$$

As $\rho \in \partial W^*$ was arbitrary, we have

$$T(f^3) = (Tf)^3$$

for all $f \in C_0(X)$, that is, $T$ is a triple homomorphism. This completes the proof. □
Remark 2. We note that the map \{u, T(\cdot), u\} in Theorem 1 is complex conjugate linear and it is equivalent to stating that the complex linear map \(P_2(u) \circ T\) is an isometry.

**Theorem 2.** Let \(T : Z \rightarrow W\) be a linear isometry between JB*-triples \(Z\) and \(W\). Then for any \(z \in Z\), there is a tripotent \(u_z \in W^{**}\) such that
\[
\{u_z, T(a^{(3)}), u_z\} = \{u_z, (Ta)^{(3)}, u_z\}
\]
for all \(a\) in the subtriple \(Z_z\) generated by \(z\), and that
\[
\{u_z, T(\cdot), u_z\} : Z_z \rightarrow W^{**}
\]
is an isometry.

**Proof.** Let \(z \in Z\). If the restriction \(T : Z_z \rightarrow W\) is a triple homomorphism, one can take \(u_z \in W^{**}\) to be the support tripotent of \(T(z)\); otherwise, Theorem 1 furnishes the required tripotent \(u_z\).

**Example 4.** Let \(T : \mathbb{C} \rightarrow M_2\) be the isometry defined in Example 1:
\[
T(a) = \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix}.
\]

Then the tripotent
\[
u = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}
\]
satisfies the conditions in Theorem 1.

In Theorem 2, the tripotent \(u_z\) depends on the given element \(z \in Z\). Extending the arguments in the proof of Theorem 1, we show below that, if \(Z\) admits a character, then one can find a tripotent \(v \in W^{**}\) such that \(\{v, T(\cdot), v\} \neq 0\) and
\[
\{v, T(a^{(3)}), v\} = \{v, (Ta)^{(3)}, v\}
\]
for all \(a \in Z\). Without any condition on \(Z\), such a tripotent \(v\) may not exist.

A character \(\varphi\) of a JB*-triple \(Z\) is a non-zero triple homomorphism \(\varphi : Z \rightarrow \mathbb{C}\).

**Lemma 2.** Let \(\varphi\) be a character of a JB*-triple \(Z\). Then \(\varphi\) is an extreme point of the closed unit ball of \(Z^*\).

**Proof.** Since \(\varphi : Z \rightarrow \mathbb{C}\) is a triple homomorphism, the induced quotient map \(\overline{\varphi} : Z/\ker \varphi \rightarrow \mathbb{C}\) is a triple isomorphism and hence an isometry. In particular, \(\|\varphi\| = 1\). Let \(e \in Z^{**}\) be the support tripotent of \(\varphi\). For \(f \in Z^*\), we denote by \(P_2(e) f\) the composite function \(f \circ P_2(e) \in Z^*\) and \(P_2(e) Z^*\) is defined accordingly.

If \(\varphi = \frac{1}{2} f + \frac{1}{2} g\) with \(\|f\|, \|g\| \leq 1\), then we have \(\|f\| = \|g\| = 1\) and
\[
1 = \|\varphi\| = \|P_2(e) \varphi\| \leq \frac{1}{2} \|P_2(e) f\| + \frac{1}{2} \|P_2(e) g\| \leq 1
\]
which yields \( \| P(e) f \| = \| P(e) g \| \) and so \( P(e) f = f \) and \( P(e) g = g \) by [9, Proposition 1]. It follows that \( \varphi \) is an extreme point of the unit ball of \( Z^* \) if, and only if, it is an extreme point of the unit ball of \( P(e) Z^* \).

Now consider the character \( \varphi : P(e) Z^{**} \to \mathbb{C} \) as a weak* continuous functional. The kernel \( \ker \varphi \) is a weak*-closed Jordan ideal in the JBW*-algebra \( (P(e) Z^{**}, \circ) \). Hence there is a central projection \( q \) in \( P(e) Z^{**} \) such that \( \ker \varphi = P(e) Z^{**} \circ q \) [10, 4.3.6]. The projection \( e - q \) has a weak*-closed support face in \( P(e) Z^* \), namely,

\[
F_{e - q} = \{ \psi \in P(e) Z^* : \| \psi \| = \psi(e) = 1 = \psi(e - q) \}.
\]

Pick an extreme point \( \rho \) from \( F_{e - q} \). Then \( \rho(\ker \varphi) = \{0\} \) implies that \( \varphi = \rho \) which is an extreme point of the unit ball of \( P(e) Z^* \).

**Proposition 2.** Let \( T : Z \to W \) be a linear isometry between JB*-triples. If \( Z \) admits a character, then there is a tripotent \( u \) in \( W^{**} \) such that \( \{ u, T(\cdot), u \} : Z \to W^{**} \) is a nonzero triple homomorphism and

\[
\{ u, T(a(3)), u \} = \{ u, (Ta)(3), u \} \quad (a \in Z).
\]

**Proof.** Let \( \eta \) be a character of \( Z \) and consider the isometry \( T^* : T(Z)^* \to Z^* \). Since \( \eta \) is the pre-image of an extreme point of the unit ball of \( T(Z)^* \), and since the extreme points in the unit ball of \( T(Z)^* \) can be extended to the extreme points in the unit ball of \( W^* \), we see that there is an extreme point \( \varphi \) of the unit ball of \( W^* \) such that \( \varphi \circ T = \eta \). Let \( u \in W^{**} \) be the minimal tripotent supporting \( \varphi \). Then

\[
\{ u, T(\cdot), u \} = \varphi \circ T(\cdot)u = \eta(\cdot)u
\]

implies that \( \{ u, T(\cdot), u \} \) is a nonzero triple homomorphism, and as in the proof of Theorem 1, we have

\[
\{ u, T(a(3)), u \} = \{ u, (Ta)(3), u \} \quad (a \in Z).
\]

The converse of Proposition 2 holds if \( W \) is abelian.

**Proposition 3.** Let \( T : Z \to W \) be a linear isometry between JB*-triples where \( W \) is an abelian C*-algebra. The following conditions are equivalent:

(i) there is a tripotent \( u \in W^{**} \) such that \( \{ u, T(\cdot), u \} \neq 0 \) and \( \{ u, T(a(3)), u \} = \{ u, (Ta)(3), u \} \) for \( a \in Z \);

(ii) \( Z \) admits a character.

**Proof.** Let \( u \) be the tripotent in (i) such that \( \{ u, T(\cdot), u \} \neq 0 \). Then there exists a character \( \rho \) of \( W \) which does not vanish on \( \{ u, T(Z), u \} \), and hence the composite \( \rho \circ \{ u, T(\cdot), u \} : Z \to \mathbb{C} \) is a non-zero triple homomorphism.

**Example 5.** Let \( T : M_2 \to C(Y) \) be the natural linear isometry into the continuous functions on the closed unit ball \( Y \) of \( M_2^* \). Since \( M_2 \) has no character, there is no tripotent in \( C(Y)^{**} \) satisfying Proposition 2.
5. Isometries in JB*-algebras

In this section, we consider a linear isometry from a JB*-triple into a JB*-algebra. This is motivated by the fact that, given a linear isometry $T : Z \to W$ between JB*-triples, by considering the second dual map, we may assume that $W$ is a JBW*-triple which is, via an isometric embedding [5], a subtriple of a JBW*-algebra. This leads to the case in which the range $W$ can be taken as a JB*-algebra. We will prove a more general result for linear contractions from JB*-triples into JB*-algebras. In this case, they may still preserve a fair amount of Jordan structure, after scaling down by a projection.

We first need to develop some basic results for JB*-algebras in which one can make good use of projections apart from tripotents. The Jordan product in a JB*-algebra will be denoted by $\circ$. We note that every JBW*-algebra $A$ has an identity $1$ [10, 4.1.7] and a continuous linear functional $\phi$ on $A$ is positive if, and only if, $\|\phi\| = \phi(1)$. If $\phi$ is a positive functional and if $\phi(p) = \phi(1)$ for some projection $p$ in $A$, then we have

$$\phi(a \circ p) = \phi(a) \quad (a \in A).$$

Indeed, if $a = a^*$, then the Schwarz inequality [10, 3.6.2] gives

$$0 \leq \phi(a \circ (1 - p))^2 \leq \phi(a^2)\phi((1 - p)^2) = 0$$

and therefore $\phi(a \circ (1 - p)) = 0$. We also have

$$\phi[p, a, p] = \phi(2p \circ (p \circ a) - p \circ a) = \phi(a).$$

Let $\phi$ be a normal state of $A$. Since the projections in $A$ form a complete lattice [10, 4.2.8], there is a smallest projection $p_\phi \in A$ such that $\phi(p_\phi) = 1$. We call $p_\phi$ the support projection of $\phi$. For any positive normal functional $\phi$, its support projection is the smallest projection $p_\phi$ in $A$ satisfying $\phi(p_\phi) = \phi(1)$. More generally, a norm-closed face of the normal state space of $A$ also admits a support projection, shown in the following lemma.

Lemma 3. Let $F$ be a norm-closed face of the normal state space $S$ of a JBW*-algebra $A$. Then there is a projection $p \in A$ such that

$$F = \{\phi \in S : \phi(p) = 1\}.$$ 

Proof. Since $F$ is a norm-closed face of the closed unit ball of the predual $A_*$ of $A$, it follows from [8, Corollary 4.5] that $F$ is a norm-exposed face of $S$. By [1], every norm-exposed face of $S$ is of the above form. \qed

Given a JB*-algebra $A$, we let

$$Q(A) = \{\phi \in A^* : \phi \geq 0 \text{ and } \|\phi\| \leq 1\}$$

be the quasi-state space of $A$. Given a projection $p$ in $A^{**}$, the set

$$F^+(p) = \{\phi \in Q(A) : \phi(1 - p) = 0\}$$

is a face of $Q(A)$ containing $0$. We show below that all weak* closed faces of $Q(A)$ containing $0$ are of this form.
Lemma 4. Let $A$ be a JB*-algebra and let $F \subset Q(A)$ be a weak* closed face of $Q(A)$ containing 0. Then there is a projection $p$ in $A^{**}$ such that

$$F = F^+(p) = \{ \varphi \in Q(A) : \varphi(1 - p) = 0 \}.$$ 

Proof. Let $S = \{ \varphi \in A^* : \varphi(1) = 1 = \|\varphi\| \}$ be the normal state space of $A^{**}$. We have $F = \operatorname{co}(F' \cup \{0\})$ where $F' = F \cap S$ is a weak* closed face of $S$ and by Lemma 3, there is a projection $p \in A^{**}$ such that

$$F' = \{ \varphi \in S : \varphi(p) = 1 \}$$

and it follows that $F = F^+(p)$.

Lemma 5. Let $A$ be a JC*-algebra and let $p \in A^{**}$ be a projection. Then for all $x \in A$, we have $x \circ p = 0$ if, and only if, $\varphi(x^* \circ x) = 0$ for all $\varphi \in F^+(p)$.

Proof. The second dual $A^{**}$ is a JW*-algebra and we may assume that it is a unital Jordan subalgebra of a von Neumann algebra $A$, with the same identity. Let $\varphi \in F^+(p)$. Then $\varphi(p) = \varphi(1)$ and by previous remarks, we have $\varphi(x) = \varphi(p \circ x) = \varphi((p, x, p))$ for all $x \in A$. The condition $0 = x \circ p = xp + px$ implies that $pxp = -px = -xp$ and so $px = xp = 0$. Hence $\varphi(x^* \circ x) = \varphi((p, x^* \circ x, p)) = \frac{1}{2}\varphi(p(x^* + x)x^*)p = \frac{1}{2}\varphi(0) = 0$.

For the converse, choose $\psi \in Q(A)$ and let $\widetilde{\psi}$ be a norm-preserving extension of $\psi$ to $A$. Then $\widetilde{\psi}$ is positive on $A$. Define $\varphi(\cdot) = \psi(p, (\cdot)^*, p)$. Then $\varphi \in F^+(p)$ and so $\varphi(x^* \circ x) = 0$. The Schwarz inequality gives

$$|\widetilde{\psi}(px)|^2 + |\widetilde{\psi}(xp)|^2 \leq \widetilde{\psi}(pxx^*p) + \widetilde{\psi}(px^*xp) = 2\widetilde{\psi}(p(x^* \circ x)p)$$

$$= 2\psi(p, (x^* \circ x), p) = 2\psi(x^* \circ x) = 0.$$ 

Hence $\widetilde{\psi}(px) = \widetilde{\psi}(xp) = 0$ and $\psi(x \circ p) = \widetilde{\psi}(x \circ p) = 0$. As $\psi$ was arbitrary in $Q(A)$, it follows that $x \circ p = 0$.

Proposition 4. Let $B$ be a JB*-algebra and let $p \in B^{**}$ be a projection. Then for $x \in B$, the following conditions are equivalent:

(i) $x \circ p = 0$;
(ii) $\varphi(x^* \circ x) = 0$ for all $\varphi \in F^+(p)$.

Proof. Let $B_{sa}$ be the self-adjoint part of $B$. First, let $x \in B_{sa}$ and let $A$ be the JBW*-subalgebra of $B^{**}$ generated by $x$, $p$ and $1$. Then $A$ is a JW*-algebra and by Lemma 5, we have $x \circ p = 0$ if, and only if, $\psi(x^2) = 0$ for all $\psi \in F^+_A(p)$, where

$$F^+_A(p) = \{ \psi \in A_* : \psi(1) = \|\psi\| \text{ and } \psi(1 - p) = 0 \}.$$ 

Since every $\varphi \in F^+(p)$ restricts to a quasi-state $\varphi|_A \in F^+_A(p)$ and since every $\psi \in F^+_A(p)$ extends to a quasi-state $\psi \in F^+(p)$, we have $x \circ p = 0$ if, and only if, $\varphi(x^2) = 0$ for all $\varphi \in F^+(p)$.

Now for any $x \in B$, write $x = x_1 + ix_2$ with $x_1, x_2 \in B_{sa}$. Then $x \circ p = 0$ if, and only if, $x_1 \circ p = 0$ and $x_2 \circ p = 0$. This is equivalent to $\varphi(x_1^2) = 0 = \varphi(x_2^2)$ for all $\varphi \in F^+(p)$ which is the same as $\varphi(x_1^2 + x_2^2) = 0 = \varphi(x^* \circ x)$ for every $\varphi \in F^+(p)$. 

\qed
Two self-adjoint elements $a$ and $p$ in a JB*-algebra $B$ are said to operator commute if they generate an associative subalgebra of $B$. If $p$ is a projection, this is equivalent to $\{ p, a, p \} = a \circ p$, and to $\{ a, p, a \} = a^2 \circ p$ (cf. [10, Lemma 2.5.5]). Condition (ii) below is an operator commuting condition.

**Theorem 3.** Let $Z$ be a JB*-triple, $B$ be a JB*-algebra and let $T : Z \to B$ be a linear contraction. Then there is a largest projection $p$ in $B^{**}$ such that for all $a, b, c \in Z$, we have

(i) $T \{ a, b, c \} \circ p = \{ Ta, Tb, Tc \} \circ p$;

(ii) $\{ p, T(a)^* \circ T(b), p \} = (T(a) \circ T(b))^* \circ p$.

**Proof.** Let

$$F_1 = \bigcap_{a \in Z_1} \{ \varphi \in Q(B) : \varphi \left( (Ta^{(3)} - (Ta)^{(3)})^* \circ (Ta^{(3)} - (Ta)^{(3)}) \right) = 0 \}.$$ 

Then $F_1$ is a weak* closed face of $Q(B)$ containing zero. For $a$ in $Z_1$, we define a weak* continuous affine map $\Phi_a : Q(B) \to Q(B)$ by

$$\Phi_a(\varphi)(\cdot) = \overline{\varphi([Ta]^* \circ Ta, \cdot, (Ta)^* \circ Ta)}$$

where the bar ‘−’ denotes complex conjugation. For $n = 1, 2, \ldots$, the sets

$$F_{n+1} = \{ \varphi \in F_n : \Phi_a(\varphi) \in F_n \text{ for all } a \in Z_1 \} = \bigcap_{a \in Z_1} F_n \cap \Phi_a^{-1}(F_n)$$

form a decreasing sequence of weak* closed faces of $Q(B)$. The intersection $F = \bigcap_{n=1}^\infty F_n$ is a weak* closed face of $Q(B)$ containing zero. By Lemma 4, there is a projection in $p \in B^{**}$ supporting $F$:

$$F = F^+(p) = \{ \varphi \in Q(B) : \varphi(1 - p) = 0 \}.$$

For each $a$ in $A_1$ and $\varphi$ in $F$, we have

$$\Phi_a(\varphi)(\cdot) = \overline{\varphi([Ta]^* \circ Ta, \cdot, (Ta)^* \circ (Ta)])} \in F,$$

and consequently,

$$\varphi([Ta]^* \circ Ta, p \cdot (Ta)^* \circ Ta) = \overline{\Phi_a(\varphi)(p)} = \overline{\Phi_a(\varphi)(1)} = \varphi((Ta)^* \circ Ta)^2).$$

Let $z = (Ta)^* \circ Ta$. Then $z$ is self-adjoint, as is $x = p \circ z - z$. For all $\varphi \in F^+(p)$,

$$\varphi(x^* \circ x) = \varphi((p \circ z - z)^2)$$

$$= \varphi((p \circ z)^2 - 2z \circ (p \circ z) + z^2)$$

$$= \varphi((p \circ z)^2 - [z, p, z] - z^2 \circ p + z^2)$$

$$= \varphi((p \circ z)^2 - [z, p, z]).$$
using the fact that \( \varphi(z^2) = \varphi(z^2 \circ p) \). By calculating in the special subalgebra generated by \( p \) and \( z \), one obtains
\[
(p \circ z)^2 = \frac{1}{2} p \circ [z, p, z] + \frac{1}{4} [p, z^2, p] + \frac{1}{4} [z, p, z].
\]
Hence we have
\[
4\varphi(x^* \circ x) = \varphi(2p \circ [z, p, z] + [p, z^2, p] + [z, p, z] - 4[z, p, z])
\]
\[
= \varphi(2[z, p, z] + z^2 - 3[z, p, z])
\]
\[
= \varphi(z^2) - \varphi([z, p, z])
\]
\[
= \Phi_a(1) - \Phi_a(p) = 0.
\]
By Proposition 4, we have \( p \circ x = 0 \). As \( p \) is a projection, it follows that \( \{p, z, p\} - p \circ z = 2(p \circ z) \circ p - 2p \circ z = 2p \circ x = 0 \), that is,
\[
\{p, (Ta)^* \circ (Ta), p\} = ((Ta)^* \circ (Ta)) \circ p
\]
for all \( a \in Z_1 \). By polarization, we have
\[
\{p, (Ta)^* \circ (Tb), p\} = ((Ta) \circ (Tb)^*) \circ p
\]
for all \( a, b \in Z \). For \( a \in Z_1 \), we have
\[
\varphi \left( (Ta^{(3)} - (Ta)^{(3)})^* \circ (Ta^{(3)} - (Ta)^{(3)}) \right) = 0
\]
for all \( \varphi \in F \), hence Proposition 4 yields
\[
(Ta^{(3)}) \circ p = (Ta)^{(3)} \circ p.
\]
Polarization then gives
\[
T[a, b, c] \circ p = [Ta, Tb, Tc] \circ p \quad (a, b, c \in Z).
\]
Finally, if \( q \) is a projection in \( B^{**} \) satisfying conditions (i) and (ii), then \( F^+(q) \subset F_1 \). Indeed, for \( \varphi \in F^+(q) \), we have
\[
\varphi \left( (Ta^{(3)} - (Ta)^{(3)})^* \circ (Ta^{(3)} - (Ta)^{(3)}) \right) = 0
\]
since \((Ta^{(3)} - (Ta)^{(3)}) \circ q = 0\) by (i) and Proposition 4 applies. Further, for all \( a \in Z_1 \), we have \( \Phi_a(F^+(q)) \subset F^+(q) \) since
\[
\Phi_a(\varphi)(q) = \varphi \left( ((Ta)^* \circ Ta, q, (Ta)^* \circ Ta) \right)
\]
\[
= \Phi_a(\varphi)(((Ta)^* \circ Ta)^2 \circ q)
\]
\[
= \Phi_a(\varphi)(((Ta)^* \circ Ta)^2) = \Phi_a(\varphi)(1)
\]
where the second identity follows from (ii). Therefore \( F^+(q) \subset \bigcap_{n=1}^{\infty} F_n = F^+(p) \) and \( q \leq p \).
Remark 3. (1) We note that condition (i) in Theorem 3 also gives
\[ p, T[a,a,a], p = 2(p \circ T[a,a,a]^*) \circ p - p \circ T[a,a,a]^* = p, [Ta,Ta,Ta], p. \]

(2) If \( B \) is a JC*-algebra in Theorem 3, then condition (i) gives
\[ T[a,a,a]p + pT[a,a,a] = [Ta,Ta,Ta]p + p[Ta,Ta,Ta] \]
and by (1) above, we have both \( T[a,a,a]p = [Ta,Ta,Ta]p \) and \( pT[a,a,a] = p[Ta,Ta,Ta]. \)

(3) If \( B \) is a JBW*-algebra in Theorem 3, then \( p \) can be chosen in \( B \) itself. Indeed, we have \( B = z \circ B^{**} \) for some central projection \( z \in B^{**} \) and \( z \circ p \) is the largest projection satisfying conditions (i) and (ii).

Example 6. The projection \( p \) in Theorem 3 could be zero, even if \( T \) is an isometry. Indeed, for the isometry \( T : \mathbb{C} \rightarrow M_2 \) in Example 1, we have \( p = 0 \). On the other hand, for the isometry \( S : \mathbb{C} \rightarrow M_3 \) given by
\[
S(a) = \begin{pmatrix} 0 & 0 & \alpha \\ 0 & a & 0 \\ \alpha & 0 & 0 \end{pmatrix}
\]
we have
\[
p = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.
\]
Moreover \( S(\cdot) \circ p \) is an isometry.

Example 7. Let \( T : C(\Omega) \rightarrow C(\Omega \cup \{\beta\}) \) be the non-surjective isometry given in Example 3. Then the characteristic function \( p = \chi_{\Omega} \in C(\Omega \cup \{\beta\}) \) is the largest projection satisfying the conclusion of Theorem 3.

References

16. Loos, O.: Bounded symmetric domains and Jordan pairs, Mathematical Lectures, University of California, Irvine, 1977