Notes on the Rigidity of Graphs

Bill Jackson*

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1 Introduction

The first reference to the rigidity of frameworks in the mathematical literature occurs in a problem posed by Euler in 1776, see [8]. Consider a polyhedron P in 3-space. We view P as a ' panel-and-hinge framework' in which the faces are 2-dimensional panels and the edges are 1-dimensional hinges. The panels are free to move continuously in 3-space, subject to the constraints that the shapes of the panels and the adjacencies between pairs of panels are preserved, and that the relative motion between pairs of adjacent panels is a rotation about their common hinge. The polyhedron P is rigid if every such motion results in a polyhedron which is congruent to P. Euler's conjecture was that every polyhedron is rigid.

The conjecture was verified for the case when P is convex by Cauchy [3] in 1813. Indeed Cauchy proved an even stronger result. Suppose P_1 and P_2 are two convex polyhedra. If there is a bijection between the faces of P_1 and P_2 which preserves both the shapes of faces and the adjacencies between pairs of faces, then P_1 and P_2 are congruent. Cauchy's strengthening of Euler's conjecture is not true for *all* polyhedra, however. Consider the icosahedron, P_1. We can reflect one of the vertices of P_1 in the plane containing it's five neighbouring vertices to obtain a non-convex polyhedron P_2 with the same faces and adjacencies between faces as P_1. Clearly P_1 and P_2 are not congruent. This example is not a counterexample to Euler's original conjecture since the reflection is not a *continous* motion from P_1 to P_2.

Gluck [9] showed in 1975 that Euler's conjecture is true when P is a 'generic' polyhedron i.e. there are no algebraic dependencies between the coordinates of the vertices of P. It follows that 'almost all' polyhedra are

*School of Mathematical Sciences, Queen Mary, University of London, Mile End Road, London E1 4NS, England. e-mail: b.jackson@qmul.ac.uk.

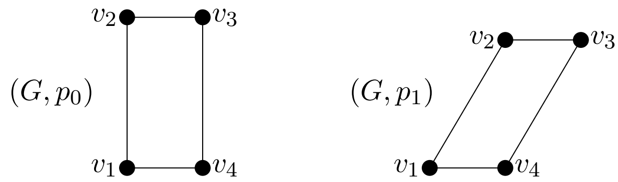

Figure 1: A 2-dimensional example. The framework (G, p_1) can be obtained from (G, p_0) by a continuous motion which preserves all edge lengths, but changes the distance between v_1 and v_3. Thus (G, p_0) is not rigid.

rigid. Connelly [4] finally showed that Euler's conjecture was false in 1982 by constructing a polyhedron which is not rigid.

We will consider a different kind of framework called a bar-and-joint framework. This is a graph G together with a map p of the vertices of G into d-space. We view the edges of (G, p) as 'bars' and the vertices as 'universal joints'. The vertices are free to move continuously, subject to the constraint that the distances between pairs of adjacent vertices are preserved i.e. the lengths of the bars does not change. The framework is rigid if every such motion preserves the distances between *all* pairs of vertices, see Figure 1. Gluck [9] showed that the rigidity of a generic bar-and-joint framework depends only on the structure of the graph G i.e. it is the same for all generic realizations of G in d-space. One of the two main problems we will consider is that of characterizing the graphs G with the property that every generic realization of G in d-space is rigid. We will see that this problem can be solved for $d = 1, 2$.

Our second main problem concerns 'global rigidity'. A d-dimensional framework (G, p) is globally rigid if every d-dimensional framework (G, q) with the same distances between pairs of adjacent vertices as (G, p), has the same distances between all pairs of vertices as (G, p), see Figure 2.

We will see that we can also characterize the graphs G with the property that every generic realization of G in d-space is globally rigid, when $d = 1, 2$. The problems of characterizing when a d-dimensional generic framework is rigid or globally rigid are unsolved for $d \geq 3$.

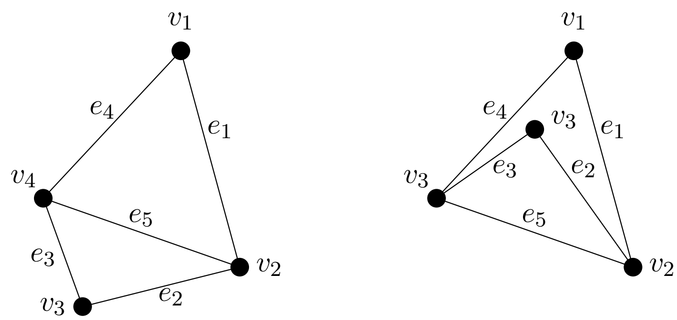

Figure 2: A rigid 2-dimensional framework which is not globally rigid. All edges in both frameworks have the same length, but the distance from v_1 to v_3 is different.

2 Bar-and-joint frameworks

A *d-dimensional bar-and-joint framework* is a pair (G, p), where $G = (V, E)$ is a graph and p is a map from V to $\mathbb{R}^d$ such that $p(u) \neq p(v)$ for all $uv \in E$. We consider the framework to be a straight line realization of G in $\mathbb{R}^d$ in which the *length* of an edge $uv \in E$ is given by the Euclidean distance between the points $p(u)$ and $p(v)$. Given frameworks (G, p_0) and (G, p_1), we say that:

- (G, p_0) and (G, p_1) are *equivalent* if $\|p_0(u) - p_0(v)\| = \|p_1(u) - p_1(v)\|$ for all $uv \in E$.
- (G, p_0) and (G, p_1) are *congruent* if $\|p_0(u) - p_0(v)\| = \|p_1(u) - p_1(v)\|$ for all $u, v \in V$.
- (G, p_0) is *globally rigid* if every framework which is equivalent to (G, p_0) is congruent to (G, p_0).
- (G, p_0) is *rigid* if there exists an $\epsilon > 0$ such that every framework (G, p_1) which is equivalent to (G, p_0) and satisfies $\|p_0(v) - p_1(v)\| < \epsilon$ for all $v \in V$, is congruent to (G, p_0). (Asimow and Roth [1, Proposition 1] show that this is equivalent to saying that every continuous motion of the vertices of (G, p_0) which preserves the lengths of all edges of (G, p_0), also preserves the distances between all pairs of vertices of (G, p_0).)

It is a difficult problem to determine whether a given d-dimensional framework (G, p) is rigid or globally rigid. We will characterize when a 1-dimensional framework is rigid in Section 3. The general feeling seems to be that it is NP-hard to determine if a d-dimensional framework is rigid for $d \geq 2$, although no proof of this is known. Saxe [21] has shown that it *is* NP-hard to determine if a d-dimensional framework is globally rigid for all $d \geq 1$. Both problems become easier, however, if we restrict our attention to generic frameworks, where a framework (G, p) is *generic* if the (multi)set containing the coordinates of all the points $p(v)$, $v \in V$, is algebraically independent over $\mathbb{Q}$. The main reason for this is that, for generic frameworks, rigidity is equivalent to the stronger property of 'infinitesimal rigidity', which we will describe below.

2.1 The rigidity matrix and infinitesimal rigidity

Let (G, p_0) and (G, p_1) be two d-dimensional frameworks. A *motion* from (G, p_0) to (G, p_1) is a function $P : [0, 1] \times V \to \mathbb{R}^d$ such that:

(M1) $P(0, v) = p_0(v)$ and $P(1, v) = p_1(v)$ for all $v \in V$;

(M2) $\|P(t, u) - P(t, v)\| = \|p_0(u) - p_0(v)\|$ for all $t \in [0, 1]$ and all $uv \in E$;

(M3) $P(t, v)$ is a continuous function of t for all $v \in V$.

Thus (G, p_0) is rigid if and only if every motion of (G, p_0) results in a framework which is congruent to (G, p_0). It can be shown using 'elementary' differential geometry, see [1], that the existence of a motion from (G, p_0) to (G, p_1) implies the existence of a smooth motion from (G, p_0) to (G, p_1) i.e. a motion P such that $P(t, v)$ is a differentiable function of t for all $v \in V$. This allows us to rewrite the equation given in (M2) as

$$\|P(t, u) - P(t, v)\|^2 = \|p_0(u) - p_0(v)\|^2,$$

and then differentiate with respect to t to obtain

$$[P(t, u) - P(t, v)] \cdot [P'(t, u) - P'(t, v)] = 0.$$

Note that $P'(t, v)$ represents the instantaneous velocity of the vertex v at time t. Putting $t = 0$ and $P'(0, v) = q(v)$ for all $v \in V$ we obtain

$$[p_0(u) - p_0(v)] \cdot [q(u) - q(v)] = 0 \text{ for all } uv \in E. \tag{1}$$

Now suppose that we are given the framework (G, p_0). We can use the system of equations (1) involving the unknowns $q(v)$, $v \in V$, to determine all possible 'instantaneous velocities' of the vertices of G which are induced by a smooth motion of (G, p_0). The *rigidity matrix* $M(G, p_0)$ of (G, p_0) is the matrix of coefficients of this system of equations. It is an $|E| \times d|V|$ matrix with rows indexed by E and sequences of d consecutive columns indexed by V. The entries in the row corresponding to an edge $e \in E$ and columns corresponding to a vertex $u \in V$ are given by the vector $p_0(u) - p_0(v)$ if $e = uv$ is incident to u and is the zero vector if e is not incident to u.

Example Let (G, p) be either of the 2-dimensional frameworks shown in Figure 2, and let $p(v_i) = (x_i, y_i)$ for $1 \leq i \leq 4$. Then the rigidity matrix of (G, p) is the matrix $M(G, p)$ shown below.

$$\begin{pmatrix} x_1 - x_2 & y_1 - y_2 & x_2 - x_1 & y_2 - y_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & x_2 - x_3 & y_2 - y_3 & x_3 - x_2 & y_3 - y_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & x_3 - x_4 & y_3 - y_4 & x_4 - x_3 & y_4 - y_3 \\ x_1 - x_4 & y_1 - y_4 & 0 & 0 & 0 & 0 & x_4 - x_1 & y_4 - y_1 \\ 0 & 0 & x_2 - x_4 & y_2 - y_4 & 0 & 0 & x_4 - x_2 & y_4 - y_2 \end{pmatrix}$$

We use $Z(G, p_0)$ to denote the null space of the matrix $M(G, p_0)$ and refer to the vectors in $Z(G, p_0)$ as *instantaneous motions* of the framework (G, p_0). We will abuse this terminology somewhat and also consider an instantaneous motion $q \in Z(G, p_0)$ as a map from V to $\mathbb{R}^d$, by indexing the components of q in the same way as the columns of $M(G, p_0)$.

For integers $n \geq 2$ and $d \geq 1$, let

$$S(n, d) = \begin{cases} dn - \binom{d+1}{2} & \text{if } n \geq d + 2 \\ \binom{n}{2} & \text{if } n \leq d + 1 \end{cases}$$

Theorem 2.1 *[1] Let (G, p) be a d-dimensional framework with $n \geq 2$ vertices. Then rank $M(G, p) \leq S(n, d)$. Furthermore, if equality holds, then (G, p) is rigid.*

Sketch Proof We consider the case when (G, p) is properly embedded in $\mathbb{R}^d$ i.e. the affine hull of the points $p(v)$, $v \in V$, is equal to $\mathbb{R}^d$. (This implies in particular that $n \geq d + 1$.) Each translation and rotation of $\mathbb{R}^d$ gives rise to a smooth motion of (G, p) and hence to an instantaneous motion of (G, p). Let $Z_0(G, p)$ be the subspace of $Z(G, p)$ generated by these special instantaneous motions. The subspace $Z_0(G, p)$ contains a linearly independent set of instantaneous motions corresponding to the translations along each vector in the standard basis, and the rotations about the $(d-2)$-dimensional subspaces containing each set of $(d-2)$ vectors in the standard basis. Thus

$$\dim Z(G,p) \geq \dim Z_0(G,p) \geq d + \binom{d}{d-2} = \binom{d+1}{2} \qquad (2)$$

and hence rank $M(G,p) \leq dn - \binom{d+1}{2}$.

To indicate why the second part of the theorem holds we suppose that (G,p) is not rigid. One may use the definition of rigidity to show that this assumption will imply that there exists a smooth motion $P(t,v)$ of (G,p) such that $\|P(t,x)\| - \|P(t,y)\|^2 \neq \|p(x)-p(y)\|^2$ for all $t > 0$ and some fixed $x, y \in V$. Differentiating with respect to t and putting $t = 0$ we deduce that there exists an instantaneous motion q of (G,p) such that $[p(x)-p(y)] \cdot [q(x)-q(y)] \neq 0$. Since translations and rotations preserve distances between all points of $\mathbb{R}^d$, $q \notin Z_0(G,p)$. Thus strict inequality must occur in (2) and rank $M(G,p) < dn - \binom{d+1}{2}$. •

We say that (G,p) is *infinitesimally rigid* if rank $M(G,p) = S(n,d)$. Theorem 2.1 implies that the infinitesimally rigidity of a given framework (G,p) is a sufficient condition for the rigidity of (G,p). The example shown in Figure 3 shows that infinitesimal rigidity is not equivalent to rigidity.

On the other hand, Asimow and Roth showed that infinitesimal rigidity is equivalent to rigidity for generic frameworks.

Theorem 2.2 *[1] Let (G,p) be a d-dimensional generic framework. Then (G,p) is rigid if and only if (G,p) is infinitesimally rigid.*

It is easy to see that, for any given graph G, rank $M(G,p)$ will be maximized when (G,p) is generic, and hence that rank $M(G,p)$ is the same for all generic realizations (G,p) of G in $\mathbb{R}^d$. We denote this maximum value of rank $M(G,p)$ by $r_d(G)$, or simply $r(G)$ when the dimension is obvious from the context. Theorem 2.2 implies that a generic d-dimensional framework (G,p) is rigid if and only if $r(G) = S(n,d)$. At first sight, one may think that this could be used to determine when a given graph G has a generic rigid realization in $\mathbb{R}^d$. This is not the case since it requires us to determine the 'generic rank' of the matrix $M(G,p)$ i.e. compute rank $M(G,p)$ when the coordinates of the vectors $p(v)$, $v \in V$, are indeterminates. As noted above, this can be done when $d \in \{1,2\}$, but is an open problem for $d \geq 3$.

Let $G = (V,E)$ be a graph. We say that: G is *rigid in* $\mathbb{R}^d$ if $r_d(G) = S(n,d)$; G is *independent in* $\mathbb{R}^d$ if $r_d(G) = |E|$ i.e. the rows of a generic

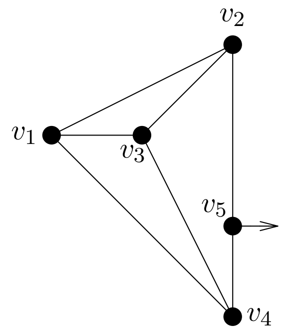

Figure 3: A 2-dimensional framework (G, p) which is rigid but not infinitesimally rigid. Let $q(v_5)$ be a vector orthogonal to the line through $p(v_2), p(v_4)$, and $q(v_i) = 0$ for $1 \leq i \leq 4$. Then $[p(v_i) - p(v_j)] \cdot [q(v_i) - q(v_j)] = 0$ for all edges v_iv_j of G, and hence q is an instantaneous motion of (G, p). However $[p(v_3) - p(v_5)] \cdot [q(v_3) - q(v_5)] \neq 0$ so $q \notin Z_0(G, p)$. It follows that $\dim Z(G, p) > \binom{2+1}{2} = 3$. Thus rank $M(G, p) < S(5, 2) = 10 - 3 = 7$, and (G, p) is not infinitesimally rigid. To see that (G, p) is rigid, suppose that there is a motion from (G, p) to another framework $(G, \tilde{p})$ and let $H = G - v_5$. It can be seen that H is rigid (since each triangle is rigid). Thus $(H, p|_H)$ is congruent to $(H, \tilde{p}|_H)$. Applying a suitable translation and rotation to $(G, \tilde{p})$ we may assume that $p(v_i) = \tilde{p}(v_i)$ for all $1 \leq i \leq 4$. The position of $p(v_5)$ on the line through $p(v_2)$ and $p(v_4)$ now implies that we must also have $p(v_5) = \tilde{p}(v_5)$.

rigidity matrix of G are linearly independent; G is *minimally rigid* if it both rigid and independent i.e. $r_d(G) = S(n, d) = |E|$. Note that if we can characterize when graphs are independent in $\mathbb{R}^d$, then we can use this to determine $r_d(G)$ for any given graph G. We greedily construct a maximal independent subgraph H of G. Then $E(H)$ will correspond to a maximal set of linearly independent rows of $M(G, p)$ in any generic realization (G, p) of G. Hence $r_d(G) = |E(H)|$. In particular G will be rigid in $\mathbb{R}^d$ if and only if $|E(H)| = S(n, d)$ i.e. H is minimally rigid.

We can use Theorem 2.1 to obtain a necessary condition on a graph $G = (V, E)$ for the rigidity matrix of a (generic) realization of G to have independent rows. For $X \subseteq E$ let $E_G(X)$ denote the set, and $i_G(X)$ the number, of edges in the subgraph of G induced by X. We will suppress the subscript and refer simply to $E(X)$ and $i(X)$ when it is obvious which graph we are refering to.

Lemma 2.3 *Let (G, p) be a d-dimensional framework. Suppose the rows of $M(G, p)$ are linearly independent. Then $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d + 2$.*

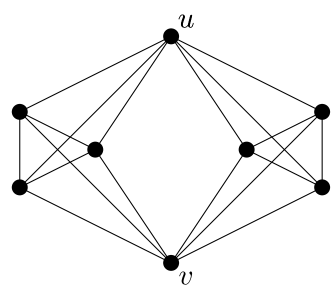

Figure 4: A 3-dimensional example. The above graph $G = (V, E)$ satisfies the condition that $i(X) \leq 3|X| - 6$ for all $X \subseteq V$ with $|X| \geq 4$. Let $M(G, p)$ be the rigidity matrix of a 3-dimensional generic framework (G, p). If the rows of $M(G, p)$ were independent, then we would have rank $M(G, p) = |E| = 3|V| - 6$. Thus G would be rigid by Theorem 2.1. Clearly this is not the case since we may rotate the right hand copy of $K_5 - e$ about the line through $p(u), p(v)$, while keeping the left hand copy fixed.

Proof: Suppose $i(X) > d|X| - \binom{d+1}{2}$ for some $X \subseteq V$ with $|X| \geq d + 1$. Let H be the subgraph of G induced by X and M_X be the submatrix of M indexed by $E(X)$ and X. Then $M_X = M(H, p|_H)$. By Theorem 2.1, rank $M_X \leq d|X| - \binom{d+1}{2}$. Since M_X has $i(X) > d|X| - \binom{d+1}{2}$ rows, the rows of M_X are linearly dependent. Since all non-zero entries in the rows of $M(G, p)$ indexed by $E(X)$ occur in the columns indexed by X, the rows of $M(G, p)$ indexed by $E(X)$ are linearly dependent. •

Note that Theorem 2.3 actually implies that $i(X) \leq S(|X|, d)$ for all $X \subseteq V$ with $|X| \geq 2$. This condition holds trivially, however, when $|X| \leq d + 1$.

We will see that the property that $i(X) \leq d|X| - \binom{d+1}{2}$ for all $X \subseteq V$ with $|X| \geq d + 2$ is sufficient to imply the linear independence of the rows of $M(G, p)$ for all 1-dimensional frameworks, and all 2-dimensional generic frameworks. It is not sufficient even for generic frameworks when $d \geq 3$. A 3-dimensional example is given in Figure 4.

3 1-dimensional frameworks

We will characterize when a 1-dimensional framework is rigid and when a 1-dimensional generic framework is globally rigid.

3.1 Rigidity

We use the following operation. Let G and H be graphs. If $H = G - v$ for some vertex v of degree one in G, we say that G is a (1-dimensional) 0-*extension* of H.

Lemma 3.1 *Let (G,p) and (H,p_H) be two 1-dimensional frameworks. Suppose G is a 0-extension of H and that p_H is the restriction of p to $V(H)$. Then the rows of $M(G,p)$ are linearly independent if and only if the rows of $M(H,p_H)$ are linearly independent.*

Proof: Let $e = vw$ be the edge of G incident to v. We may assume that e indexes the first row, and v the first column, of $M(G,p)$. The lemma now follows since $p(u) - p(v) \neq 0$ and

$$M(G,p) = \begin{pmatrix} p(u) - p(v) & * \\ 0 & M(H, p|_H) \end{pmatrix}$$

•

Theorem 3.2 *Let (G,p) be a 1-dimensional framework. Then:*
(a) the rows of $M(G,p)$ are linearly independent if and only if G is a forest;
(b) (G,p) is infinitesimally rigid if and only if G is connected;
(c) (G,p) is rigid if and only if G is connected.

Proof: (a) Suppose G is a forest. Then we can show that the rows of $M(G,p)$ are independent by induction on $|E(G)|$ using Lemma 3.1. On the other hand, if G contains a cycle C then $|E(C)| > |V(C)| - 1$ and hence the rows of $M(G,p)$ are linearly dependent by Lemma 2.3.

(b) Let $F \subseteq E(G)$ correspond to a maximal set of linearly independent rows of $M(G,p)$ and H be the spanning subgraph of G with edge set F. By (a), H is a spanning forest of G. Thus (G,p) is infinitesimally rigid if and only if rank $M(G,p) = |E(H)| = S(n,1) = n - 1$ i.e. H is a spanning tree of G.

(c) If G is connected then (G,p) is infinitesimally rigid by (b) and hence is rigid by Theorem 2.1. On the other hand, if G is disconnected then it is easy to define a motion of (G,p) which changes the distance between two vertices belonging to different components of G (we just translate one component along the real line and keep the other components fixed). •

Note that a graph $G = (V,E)$ is a forest if and only if $i(X) \leq |X| - 1$ for all $X \subseteq V$ with $|X| \geq 3$. Thus the necessary condition for the linear independence of the rows of a rigidty matrix given in Lemma 2.3 is also sufficient when $d = 1$.

Exercise Give a direct proof that (G, p) is rigid if G is connected, using the definition of rigidity given at the beginning of Section 2.

Taking the special case of Theorem 3.2 when (G, p) is generic, we immediately obtain

Corollary 3.3 *Let G be a graph. Then:*
(a) G is independent in $\mathbb{R}$ if and only if G is a forest;
(b) G is minimally rigid in $\mathbb{R}$ if and only if G is a tree;
(c) G is rigid in $\mathbb{R}$ if and only if G is connected.

3.2 Global rigidity

We shall show that a 1-dimensional generic framework (G, p) is globally rigid if and only if either G is a complete graph on at most two vertices or G is 2-connected. [1] It is easy to see that 2-connectivity is a necessary condition for the global rigidity of any framework.

Lemma 3.4 *Suppose (G, p) is a globally rigid 1-dimensional framework. Then either $G = K_2$ or G is 2-connected.*

Proof: Since (G, p) is gobally rigid, (G, p) is rigid and hence G is connected. If G has two vertices then $G = K_2$. Hence we may assume that G has at least three vertices. Suppose G is not 2-connected. Then there exists a vertex v, and subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $E(G_1) \cap E(G_2) = \emptyset$, and $V(G_1) \cap V(G_2) = \{v\}$. Applying a suitable translation along the real line, we may suppose that $p(v) = 0$. Let (G, p_1) be the framework obtained by putting $p_1(u) = p(u)$ for $u \in V(G_1)$ and $p_1(u) = -p(u)$ for $u \in V(G_2)$. (Thus (G, p_1) is obtained from (G, p) by reflecting G_2 in the point $p(v) = 0$.) Then (G, p_1) is equivalent, but not congruent, to (G, p). This contradicts the hypothesis that (G, p) is globally rigid. •

It is not true that the 2-connectivity of G is a sufficient condition for the global rigidity of a non-generic framework (G, p), see Figure 5. Indeed Saxe [21] has shown that it is NP-hard to determine whether an arbitrary framework is globally rigid. It follows that any proof of the above mentioned

[1] A simpler proof of this result, and a characterization of graphs which can be realised as 1-dimensional frameworks which are not globally rigid, were obtained by participants of the Levico meeting. They are given as appendices to these notes.

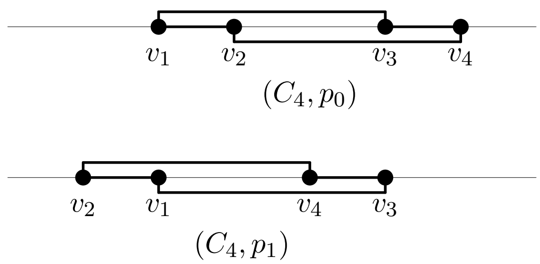

Figure 5: Two equivalent but non-congruent realizations of the 4-cycle C_4 on the real line. The frameworks are not congruent since the distance from v_1 to v_4 is different in each framework. The frameworks (C_4, p_i) are not generic since $[p_i(v_1) - p_i(v_2)]^2 = [p_i(v_3) - p_i(v_4)]^2$ for each $i \in \{1, 2\}$.

characterization of globally rigid generic frameworks must use the fact that the there are no algebraic dependencies between the numbers $p(v)$, $v \in V(G)$.

A 1-dimensional framework (G, p) is *quasi-generic* if $p(v_0) = 0$ for some $v_0 \in V$, and the (multi)set $\{p(v) \ : \ v \in V - v_0\}$ is algebraically independent over $\mathbb{Q}$. Note that every 1-dimensional generic framework can be transformed into a quasi-generic framework by a suitable translation along the real line. This transformation will not change the global rigidity of the framework. We will need the following result for equivalent quasi-generic frameworks. An analogous result for 2-dimensional frameworks was proved, using 'elementary' algebraic number theory, in [17]. The 1-dimensional result can be proved similarly.

Lemma 3.5 *Let (G, p_1) be an infinitesimally rigid 1-dimensional quasi-generic framework with $p_1(v_0) = 0$. Suppose that (G, p_2) is a 1-dimensional framework which is equivalent to (G, p_1) and has $p_2(v_0) = 0$. Then (G, p_2) is quasi-generic. Furthermore, if $\mathbb{K}_i$ is the algebraic closure of $\mathbb{Q} \cup \{p(v) \ : \ v \in V\}$ for $i = 1, 2$, then $\mathbb{K}_1 = \mathbb{K}_2$.*

We also need another graph operation. Let G and H be graphs. Suppose $H = G - v + uw$ for some vertex v of degree two in G with non-adjacent neighbours u, w. Then we say that G is a (1-dimensional) 1-*extension* of H, see Figure 6. Our next result shows that 1-extensions preserve global rigidity. Its proof is similar to that of the analogous 2-dimensional result given in [17].

Theorem 3.6 *Let (G, p) and (H, p_H) be two 1-dimensional generic frameworks. Suppose that H has at least three vertices, G is a 1-extension of H,*

Figure 6: G is a 1-dimensional 1-extension of H.

p_H is the restriction of p to $V(H)$, and (H, p_H) is globally rigid. Then (G, p) is globally rigid.

Proof: Suppose $H = G - v + uw$. Let $(G, \tilde{p})$ be an equivalent framework to (G, p). We shall show that $(G, \tilde{p})$ is congruent to (G, p).

Claim 3.7 $\|p(u) - p(w)\| = \|\tilde{p}(u) - \tilde{p}(w)\|$.

Proof: By applying a suitable translation to (G, p) and $(G, \tilde{p})$, we may assume that (G, p) is quasi-generic and $p(u) = 0 = \tilde{p}(u)$. Since (G, p) and $(G, \tilde{p})$ are equivalent, we have

$$p(v)^2 = \|p(v) - p(u)\|^2 = \|\tilde{p}(v) - \tilde{p}(u)\|^2 = \tilde{p}(v)^2 \tag{3}$$

and

$$[p(v) - p(w)]^2 = \|p(v) - p(w)\|^2 = \|\tilde{p}(v) - \tilde{p}(w)\|^2 = [\tilde{p}(v) - \tilde{p}(w)]^2. \tag{4}$$

Now (3) and (4) give

$$2\tilde{p}(v)\tilde{p}(w) = \tilde{p}(w)^2 - p(w)^2 + 2p(v)p(w).$$

Since $\tilde{p}(v)^2 = p(v)^2$ by (3), we may deduce that

$$4p(v)^2\tilde{p}(w)^2 = [\tilde{p}(w)^2 - p(w)^2 + 2p(v)p(w)]^2$$

and hence

$$[4p(v)p(w) + p(w)^2 - \tilde{p}(w)^2][\tilde{p}(w)^2 - p(w)^2] = 0. \tag{5}$$

Since (H, p_H) is globally rigid, H is 2-connected by Lemma 3.4. Thus $H - uw$ is connected and hence $(H - uw, p_H)$ is infinitesimally rigid by Theorem 3.2. Furthermore $(H - uw, p_H)$ is quasi-generic and $(H - uw, \tilde{p}_H)$

is equivalent to $(H-uw, p_H)$, since $(G,\tilde{p})$ is equivalent to (G,p). Let $\mathbb{K}$ and $\tilde{\mathbb{K}}$ be the algebraic closures of $\mathbb{Q}\cup\{p(x) : x\in V(H)\}$ and $\mathbb{Q}\cup\{\tilde{p}(x) : x\in V(H)\}$, respectively. By Lemma 3.5, $\mathbb{K}=\tilde{\mathbb{K}}$. In particular $\tilde{p}(w)\in\mathbb{K}$. Since (G,p) is quasi-generic $p(v)\notin\mathbb{K}$. Since $4p(v)p(w)+p(w)^2-\tilde{p}(w)^2$ is a polynomial in $p(v)$ with coefficients in $\mathbb{K}$, we have $4p(v)p(w)+p(w)^2-\tilde{p}(w)^2\neq 0$. Equation (5) and the fact that $p(u)=0=\tilde{p}(u)$ now imply that

$$\|p(u)-p(w)\|^2=p(w)^2=\tilde{p}(w)^2=\|\tilde{p}(u)-\tilde{p}(w)\|^2.$$

•

We can now complete the proof of the theorem. Claim 3.7 and the fact that $(H-uw,p_H)$ is equivalent to $(H-uw,\tilde{p}_H)$, imply that (H,p_H) is equivalent to $(H,\tilde{p}_H)$. Since (H,p_H) is globally rigid, (H,p_H) must be congruent to $(H,\tilde{p}_H)$. Thus there is a congruence of the real line, i.e. a translation or reflection, which maps $(H,\tilde{p}_H)$ onto (H,p_H). Since $p(v)$ is uniquely determined by $p(u)$, $p(w)$, $\|p(v)-p(u)\|$, and $\|p(v)-p(u)\|$, and since $\|p(v)-p(u)\|=\|\tilde{p}(v)-\tilde{p}(u)\|$ and $\|p(v)-p(u)\|=\|\tilde{p}(v)-\tilde{p}(u)\|$, this congruence must also map $\tilde{p}(v)$ onto $p(v)$. Thus $(G,\tilde{p})$ is congruent to (G,p). •

We can use Theorem 3.6 to deduce our characterization of globally rigid generic frameworks.

Theorem 3.8 *Let (G,p) be a 1-dimensional generic framework. Then (G,p) is globally rigid if and only if either $G=K_2$ or G is 2-connected.*

Proof: Necessity follows from Lemma 3.4. To prove sufficiency it will suffice to show that, if $G=(V,E)$ is 2-connected, then (G,p) is globally rigid. We use induction on $|E|$. If $|E|=3$ then $G=K_3$ and (G,p) is globally rigid. Thus we may suppose that $|E|\geq 4$, and hence $|V|\geq 4$. If $G-e$ is 2-connected for some $e\in E$, then $(G-e,p)$ is globally rigid by induction, and hence (G,p) is also globally rigid. Thus we may suppose that $G-e$ is not 2-connected for all $e\in E$ i.e. G is *minimally 2-connected.* It is straightforward to show that this implies the existence of a vertex v of degree two in G. Let u,w be the neighbours of v in G. The minimal 2-connectivity of G and the fact that $G\neq K_3$, imply that $uw\notin E(G)$ and $H=G-v+uw$ is 2-connected. By induction (H,p_H) is globally rigid. Hence (G,p) is globally rigid by Theorem 3.6. •

Exercise Show that every minimally 2-connected graph G contains a vertex of degree two. Show further that, if $G \neq K_3$, v is a vertex of G of degree two and and u, w are the neighbours of v, then $uw \notin E(G)$ and $G - v + uw$ is 2-connected.

Theorem 3.8 implies that global rigidity of 1-dimensional frameworks is a *generic property*, i.e. the global rigidity of a 1-dimensional framework generic (G, p) is the same for all generic realizations of G, since it depends only on the structure of the graph G. We shall see in the Section 5 that global rigidity of 2-dimensional frameworks is also a generic property. Gortler, Healy and Thurston [10] have recently shown that global rigidity of d-dimensional frameworks is a generic property for all d by solving a conjecture of Connelly. The fact that the rigidity of d-dimensional frameworks is a generic property for all d follows from Theorem 2.2 and that fact that rank $M(G, p)$ is the same for all generic d-dimensional frameworks (G, p).

4 Rigidity of generic 2-dimensional frameworks

We characterize when a graph is independent, or rigid, in $\mathbb{R}^2$.

4.1 Independence

We will prove a theorem of Laman [18] that a graph $G = (V, E)$ is independent in $\mathbb{R}^2$ if and only if $i(X) \leq 2|X| - 3$ for all $X \subset V$ with $|X| \geq 4$. We will call graphs which satisfy the latter condition *Laman* graphs. Theorem 2.3 implies that all independent graphs are Laman. To prove the reverse implication we need 2-dimensional versions of the graph operations used in Section 3. Let G and H be graphs. If $H = G - v$ for some vertex v of degree at most two, we say that G is a (2-dimensional) 0-*extension* of H. If $H = G - v + uw$ for some vertex v of degree three and non-adjacent neighbours u, w of v, then we say that G is a (2-dimensional) 1-*extension* of H, see Figure 7. These operations were first used for studying the rigidity of frameworks by Henneberg [14].

Lemma 4.1 *Let (G, p) and (H, p_H) be two 2-dimensional frameworks. Suppose G is a 0-extension of H, $p|_H$ is the restriction of p to $V(H)$, and the rows of $M(H, p|_H)$ are linearly independent. Suppose further that if $H = G - v$, $d(v) = 2$ and u, w are the neighbours of v in G, then $p(v), p(u), p(w)$ are not collinear. Then the rows of $M(G, p)$ are linearly independent.*

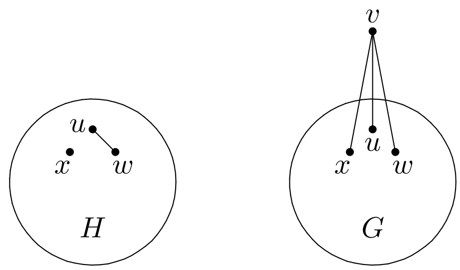

Figure 7: G is a 2-dimensional 1-extension of H.

Proof: The lemma can be proved in a similar way to Lemma 3.1. •

Lemma 4.2 *Let (G, p) and (H, p_H) be two generic 2-dimensional frameworks. Suppose G is a 1-extension of H and that $p|_H$ is the restriction of p to $V(H)$. Let $H = G - v + uw$ where u, w are non-adjacent neighbours of v in G. If the rows of $M(H, p|_H)$ are linearly independent, then the rows of $M(G, p)$ are linearly independent.*

Proof: We first construct a non-generic realization $(G + uw, \tilde{p})$ of $G + uw$, by putting $\tilde{p}(z) = p(z)$ for all vertices $z \neq v$, and choosing $\tilde{p}(v)$ to be a point on the line through $p(u), p(v)$ which is distinct from $p(u)$ and $p(v)$. Let x be the neighbour of v in G distinct from u, w. Since (G, p) is generic, $\tilde{p}(x)$ does not lie on the line through $\tilde{p}(u), \tilde{p}(v)$. Hence Lemma 4.1 implies that the rows of $M(G + uw - vw, \tilde{p})$ are linearly independent. On the other hand, the fact that $\tilde{p}(v)$, $\tilde{p}(u)$ and $\tilde{p}(w)$ are collinear implies that the rows of $M(G + uw, \tilde{p})$ indexed by vu, vw, uw are a minimally dependent set of rows of $M(G + uw, \tilde{p})$. (We can consider the submatrix induced by these rows as the rigidity matrix of a 1-dimensional realization of K_3.) This implies that we may delete any one of these rows without reducing the rank of $M(G + uw, \tilde{p})$. Thus

$$\text{rank } M(G, \tilde{p}) = \text{rank } M(G + uw - vw, \tilde{p}) = |E(G)|.$$

Hence the rows of $M(G, \tilde{p})$ are linearly independent.

We can complete the proof by noting that, since (G, p) is generic, we have rank $M(G, p) \geq$ rank $M(G, \tilde{p})$. Hence the rows of $M(G, p)$ are linearly independent. •

Exercise Extend the statement and proof of Lemmas 4.1 and 4.2 to d-dimensions

We next show that every Laman graph on at least three vertices can be obtained from a smaller Laman graph by a 0-extension or a 1-extension. Once we have this, we can deduce that all Laman graphs are independent by using Lemmas 4.1, 4.2 and induction.

Let $G = (V, E)$ be a graph. We shall need the following supermodular inequalities for the set function $i(X)$, which are easy to check by counting the contribution of an edge to each of their two sides.

Lemma 4.3 *Let G be a graph and $X, Y \subseteq V(G)$. Then*

$$i(X) + i(Y) \leq i(X \cup Y) + i(X \cap Y).$$

Lemma 4.4 *Let G be a graph and $X, Y, Z \subseteq V(G)$. Then*

$$i(X)+i(Y)+i(Z) \leq i(X \cup Y \cup Z)+i(X \cap Y)+i(X \cap Z)+i(Y \cap Z)-i(X \cap Y \cap Z).$$

Given a Laman graph $G = (V, E)$, we say that a set $X \subseteq V$ is *critical* if $i(X) = 2|X| - 3$ holds.

Lemma 4.5 *Let $G = (V, E)$ be Laman and let $X, Y \subset V$ be critical sets in G with $|X \cap Y| \geq 2$. Then $X \cap Y$ and $X \cup Y$ are both critical.*

Proof: Lemma 4.3 gives

$$\begin{aligned} 2|X| - 3 + 2|Y| - 3 &= i(X) + i(Y) \\ &\leq i(X \cap Y) + i(X \cup Y) \\ &\leq 2|X \cap Y| - 3 + 2|X \cup Y| - 3 \\ &= 2|X| - 3 + 2|Y| - 3 \end{aligned}$$

Thus equality holds everywhere and $X \cap Y$ and $X \cup Y$ are also critical. •

Lemma 4.6 *Let $G = (V, E)$ be a Laman graph. Let $X, Y, Z \subset V$ be critical sets in G with $|X \cap Y| = |X \cap Z| = |Y \cap Z| = 1$ and $X \cap Y \cap Z = \emptyset$. Then $X \cup Y \cup Z$ is critical.*

Proof: Lemma 4.4 gives

$$\begin{aligned} 2|X|-3+2|Y|-3+2|Z|-3 &\leq i(X)+i(Y)+i(Z) \\ &\leq i(X\cup Y\cup Z) \\ &\leq 2(|X\cup Y\cup Z|)-3 \\ &= 2(|X|+|Y|+|Z|-3)-3 \\ &= 2|X|-3+2|Y|-3+2|Z|-3. \end{aligned}$$

Hence equality holds everywhere and $X\cup Y\cup Z$ is critical. •

Lemma 4.7 *Let $G=(V,E)$ be a Laman graph with at least two vertices and $v\in V$.*
(a) If $d(v)\leq 2$ then $G-v$ is Laman.
(b) If $d(v)=3$ then $G-v+uw$ is Laman for some pair of non-adjacent neighbours u,w of v.

Proof: Part (a) follows easily from the definition of Laman graphs. To prove (b) we proceed by contradiction. Suppose (b) is false and let u,w,z be the neighbours of v in G. Since the (multi)graph $G-v+uw$ is not Laman, there exists a critical set $X\subset V$ with $u,w\in X$ and $v,z\notin X$. Applying the same argument to uz and vz, we deduce that there exist maximal critical sets $X_{uw},X_{uz},X_{wz}\subset V-v$ each containing precisely two neighbours ($\{u,w\},\{u,z\},\{w,z\}$, respectively) of v. By Lemma 4.5 and the maximality of these sets we must have $|X_{uw}\cap X_{uz}|=|X_{uw}\cap X_{wz}|=|X_{uz}\cap X_{wz}|=1$. Thus, by Lemma 4.6 the set $Y=X_{uw}\cup X_{uz}\cup X_{wz}$ is also critical. Since $d(v,Y)\geq 3$, this gives $i_G(Y\cup\{v\})>2|Y\cup\{v\}|-3$, which contradicts the hypothesis that G is Laman. •

We can now prove Laman's theorem.

Theorem 4.8 *[18] Let $G=(V,E)$ be a graph. Then G is independent in 2-dimensions if and only if $i(X)\leq 2|X|-3$ for all $X\subseteq V$ with $|X|\geq 4$.*

Proof: Necessity follows from Lemma 2.3. To prove sufficiency we suppose that G is Laman and choose a generic realization $M(G,p)$ of G. Since G is Laman, $|E|\leq 2|V|-3$. Thus G has a vertex of degree at most three. By Lemma 4.7, G is either a 0-extension or a 1-extension of a smaller Laman graph H. By induction $(H,p|_H)$ is independent. Hence (G,p) is independent by Lemmas 4.1 and 4.2. •

This immediately gives the following characterization of graphs which are minimally rigid in $\mathbb{R}^2$.

Corollary 4.9 *[18] A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if $|E| = 2|V| - 3$ and $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $|X| \geq 4$.*

We may also use the results of this subsection to obtain the following recursive construction for minimally rigid graphs.

Corollary 4.10 *[14, 18] A graph $G = (V, E)$ is minimally rigid in $\mathbb{R}^2$ if and only if it can be constructed from K_2 by a sequence of 0-extensions (which add a vertex of degree exactly two) and 1-extensions.*

The sequence of extensions used in Corollary 4.10 is called a *Henneberg sequence* for G.

Corollaries 4.9 and 4.10 combine to give a good characterization of minimally rigid graphs: to convince someone that a graph $G = (V, E)$ is minimally rigid we can show them a Henneberg sequence for G; to convince someone G is not minimally rigid we show that $|E| \neq 2|V| - 3$ or we show a subset $X \subseteq V$ with $|X| \geq 4$ and $i(X) > 2|X| - 3$.

4.2 Rigidity

We extend the characterization of minimally rigid graphs given in the previous subsection to a characterization of rigid graphs. This will follow from the following more general result of Lovász and Yemini [19], which determines the rank of the rigidity matrix of a generic 2-dimensional realization of a graph.

Given a graph $G = (V, E)$, a *cover* of G is a family $\mathcal{X}$ of subsets of V such that $|X| \geq 2$ for all $X \in \mathcal{X}$ and $\bigcup_{X \in \mathcal{X}} E(X) = E$. A cover $\mathcal{X}$ is 1-*thin* if $|X_i \cap X_j| \leq 1$ for all $X_i, X_j \in \mathcal{X}$.

Theorem 4.11 *[19] Let $G = (V, E)$ be a graph. Then*

$$r_2(G) = \min_{\mathcal{X}} \{ \sum_{X \in \mathcal{X}} (2|X| - 3) \}$$

where the minimum is taken over all 1-thin covers of G.

Proof: Let (G, p) be a generic realization of G in 2-dimensions and $F \subseteq E$ be the edges corresponding to a maximum set of linearly independent rows of $M(G, p)$. Since the rows indexed by F are linearly independent, we have

$|F \cap E_G(X)| \leq 2|X| - 3$ for all $X \subset V$ with $|X| \geq 2$ by Lemma 2.3. Thus $r_2(G) = |F| \leq \sum_{X \in \mathcal{X}}(2|X| - 3)$ for every 1-thin cover $\mathcal{X}$ of G.

To see that equality can be attained, let $H = (V, F)$. Consider the maximal critical sets $X_1, X_2, ..., X_t$ in H. By Lemma 4.5 we have $|X_i \cap X_j| \leq 1$ for all $1 \leq i < j \leq t$. Since every single edge of F induces a critical set, it follows that $\{E_H(X_1), E_H(X_2), ..., E_H(X_t)\}$ is a partition of F. Thus

$$r_2(G) = |F| = \sum_{i=1}^{t} |E_H(X_i)| = \sum_{i=1}^{t} (2|X_i| - 3).$$

To complete the proof we show that $\{E_G(X_1), E_G(X_2), ..., E_G(X_t)\}$ is a partition of E. Choose $uv \in E - F$. Since F indexes a maximal linearly independent set of rows of $M(G, p)$, $H + uv$ is not independent. Since H is independent, Theorem 4.8 implies that there exists a set $X \subseteq V$ such that $u, v \in X$ and $i_H(X) = 2|X| - 3$. Hence X is a critical set in H. This implies that $X \subseteq X_i$, and hence $uv \in E_G(X_i)$ for some $1 \leq i \leq t$. •

Lovász and Yemini [19] used Theorem 4.11 to show that every 6-connected graph is rigid in $\mathbb{R}^2$, and note that the same proof technique will give the following slightly stronger result.

Theorem 4.12 *[19] Let G be a 6-connected graph. Then $G - \{e_1, e_2, e_3\}$ is rigid in $\mathbb{R}^2$ for all edges e_1, e_2, e_3 in G.*

Lovász and Yemini also use Theorem 4.11 to construct a family of 5-connected graphs which are not rigid in $\mathbb{R}^2$. They conjecture that every 12-connected graph is rigid in $\mathbb{R}^3$. To date no one has been able to show that there even exists a finite k such that all k-connected graphs are rigid in $\mathbb{R}^3$.

5 Global rigidity of 2-dimensional generic frameworks

We will characterize when a 2-dimensional generic framework (G, p) is globally rigid. The characterization will depend only on the structure of the graph G and hence will imply that global rigidity is a generic property in $\mathbb{R}^2$.

5.1 Hendrickson's conditions

Hendrickson [13] obtained two necessary conditions for the global rigidity of a 2-dimensional generic framework. The first is a direct analogue of Lemma 3.4 and can be proved similarly.

Lemma 5.1 *Suppose (G,p) is a globally rigid 2-dimensional generic framework. Then either G is a complete graph on at most three vertices or G is 3-connected.*

The second necessary condition uses a new concept. We say that a graph $G=(V,E)$ is *redundantly rigid* if $G-e$ is rigid for all $e \in E$.

Theorem 5.2 *[13] Suppose (G,p) is a globally rigid 2-dimensional generic framework. Then either G is a complete graph on at most three vertices or G is redundantly rigid.*

The following is a sketch of a version of Hendrickson's proof which came out of discussions with Tibor Jordán. It contains two unsubstantiated claims which can be proved in the same way as similar statements in [13]. My purpose is to give the reader some feeling for the techniques involved. We will need some additional terminology. Let $G=(V,E)$, where $V=\{v_1,v_2,\ldots,v_n\}$, $E=\{e_1,e_2,\ldots,e_m\}$ and $e_1=v_1v_2$. By applying a suitable translation and rotation we may suppose that (G,p) is in *standard position* i.e. $p(v_1)=(0,0)$ and $p(v_2)=(0,p_4)$. We consider (G,p) as a point $(p_4,p_5,\ldots,p_{2n})$ in $\mathbb{R}^{2n-3}$ where $p(v_i)=(p_{2i-1},p_{2i})$ for $3 \le i \le n$. The *rigidity map* of (G,p) is the map $f_G : \mathbb{R}^{2n-3} \to \mathbb{R}^m$ given by $f_G(p) = (\|e_1\|,\|e_2\|,\ldots,\|e_m\|)$, where, for $e_i=uv$, we have $\|e_i\| = \|p(u)-p(v)\|^2$. We will use the fact that the Jacobian, df_G, of the rigidity map, f_G, is given by $df_G = 2M(G,p)$, where $M(G,p)$ is the rigidity matrix of (G,p).

Sketch Proof of Theorem 5.2

Consider all frameworks (K_n,q) in $\mathbb{R}^2$ with $q(v_1)=(0,0)$, $q(v_2)=(0,q_4)$, $q(v_i)=(q_{2i-1},q_{2i})$ for $3 \le i \le n$, and not all the points (q_{2i-1},q_{2i}), $3 \le i \le n$, on the y-axis. Associate each such q with the point $p=(q_4,q_5,...,q_{2n}) \in \mathbb{R}^{2n-3}$, and let T be the set of all such points. Then T is an open subset of $\mathbb{R}^{2n-3}$ and hence T is a $(2n-3)$-dimensional manifold.

Let $f=f_{K_n}$ be the rigidity map of K_n. Consider f as a map from T to $\mathbb{R}^m$ (where $m=|E(K_n|)$. Since (K_n,q) is infinitesimally rigid at each $q \in T$, rank $df|_q = 2n-3$ for all $q \in T$. Hence, by the inverse function theorem, $X=f(T)$ is a $(2n-3)$-dimensional manifold.

Suppose that $G-e$ is not rigid for some edge $e=uv$ of G. Relabelling if nescessary we may assume that $e \neq v_1v_2$. Let H be an independent spanning subgraph of $G-e$ with $|E(H)| = 2n-4$. Consider all frameworks (H,q) with $q \in T$ and let f_H be the rigidity map of H. We can consider f_H as a map from T to $\mathbb{R}^{2n-4}$. Since p is a generic point in T, rank $df_H|_p = 2n-4$, so p is a regular point of f_H. In fact:

Claim 5.3 *$f_H(p)$ is a regular value of f_H.*

Now let π_H be the projection of X into $\mathbb{R}^{2n-4}$ obtained by taking the coordinates of each point $x \in X$ indexed by edges of H. Then π_H is smooth and $f_H = \pi_H \circ f$. Thus Claim 5.3 implies that $f_H(p)$ is a regular value of π_H and rank $d\pi_H|_{f(p)} = 2n-4$. By [20, page 11, Lemma 1], $C = \pi_H^{-1}(f_H(p))$ is a 1-dimensional manifold.

Claim 5.4 *C is compact.*

Since C is a compact 1-dimensional manifold, each component of C is diffeomorphic to a circle. Let C_0 be the component of C which contains $x = f(p)$. The function $\|e\|$ changes continuously as we traverse C. Since $H+e$ is rigid and p is generic, x cannot be a critical point of this function. Thus there exists another point y on C distinct from x such that $\|e\|$ is the same at x and y. Choose $q \in f^{-1}(y)$. Then $(H+e,q)$ is equivalent, but not congruent to $(H+e,p)$. The fact that $M(H,p)$ and $M(G-e)$ have the same null space implies that the distances $\|p(u)-p(v)\|$ are preserved by every motion of (H,p), for all $uv \in E(G)-e$. Since (H,q) can be obtained by a motion of (H,p), and since we also have $\|q(u)-q(v)\| = \|p(u)-p(v)\|$, we may deduce that (G,q) is equivalent, but not congruent to, (G,p). •

Hendrickson [13] actually proved the d-dimensional generalizations of Lemma 5.1 and Theorem 5.2: if (G,p) is a globally rigid d-dimensional generic framework, then G is either a complete graph on at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$. He conjectured that these conditions were also sufficient to imply the global rigidity of a d-dimensional generic framework. Connelly [5] showed that this conjecture is false for $d=3$ by constructing a generic realization of the complete bipartite graph $K_{5,5}$ which is not globally rigid. We shall see that Hendrickson's conjecture does hold when $d=2$ i.e. a 2-dimensional generic framework (G,p) is globally rigid if and only if G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid in $\mathbb{R}^2$. The basic idea behind the proof is the same as that for global rigidity in $\mathbb{R}$. The first step is to show that global rigidity of generic frameworks is preserved by 1-extensions. The second step is to show that if G is a 3-connected graph on at least five vertices which is redundantly rigid in $\mathbb{R}^2$, then G can be obtained from a smaller 3-connected redundantly rigid graph by a 1-extension or an edge addition.

5.2 Extending globally rigid frameworks

We need the following result. Its proof is similar, but slightly more complicated than that of Theorem 3.6.

Theorem 5.5 *[17] Let (G, p) and (H, p_H) be two 2-dimensional generic frameworks. Suppose that H has at least four vertices, G is a 1-extension of H, p_H is the restriction of p to $V(H)$, and (H, p_H) is globally rigid. Then (G, p) is globally rigid.*

A slightly weaker result than Theorem 5.5 was previously obtained by Connelly [6], who showed that a 2-dimensional generic framework (G, p) is globally rigid if G can be obtained from K_4 by 1-extensions. His result can also be used to obtain the above mentioned characterization of the global rigidity of 2-dimensional generic frameworks, when combined with the results of the next subsection, see [6]. Connelly's proof is less elementary than our proof of Theorem 5.5. On the other hand, his proof extends to d-dimensions and shows that a generic d-dimensional framework (G, p) is globally rigid if G can be obtained from K_{d+1} by (d-dimensional) 1-extensions. We do not know if our proof of Theorem 5.5 can be extended to d-dimensions.

5.3 Reducing 3-connected redundantly rigid graphs

The results described in this subsection correspond to the step in the characterization of the global rigidity of 1-dimensional generic frameworks which showed that every minimally 2-connected graph can be obtained from a smaller 2-connected graph by a (1-dimensional) 1-extension. They are a combination of results from Berg and Jordán [2] and Jackson and Jordán [15]. The proof is purely graph theoretic but is considerably more complicated than the analogous 1-dimensional proof. It requires several new concepts.

5.3.1 M-circuits

Given a graph $G = (V, E)$, an *M-circuit of G* is a minimal subgraph H such that the rows of $M(G, p)$ indexed by $E(H)$ are linearly dependent in any

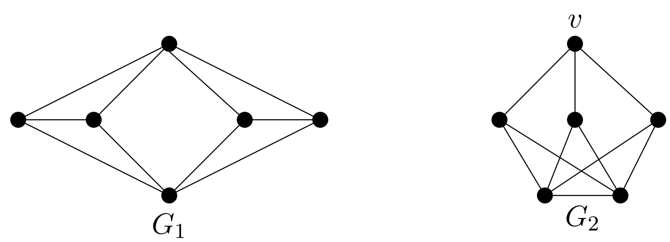

Figure 8: The M-circuit G_1 is not a 1-extension of a smaller M-circuit. The 3-connected M-circuit G_2 cannot be reduced to a smaller M-circuit by deleting v and adding an edge between two neighbours of v.

generic realization (G,p) of G. Note that

- H is an M-circuit if and only if H is not independent but every proper subgraph of H is independent, and
- G is redundantly rigid if and only if G is rigid and every edge of G belongs to an M-circuit.

The following characterization of M-circuits follows easily from Theorem 4.8.

Lemma 5.6 *Let $G = (V, E)$ be a graph. Then G is an M-circuit if and only if $|E| = 2|V| - 2$ and $i(X) \leq 2|X| - 3$ for all $X \subseteq V$ with $4 \leq |X| < |V|$.*

It follows easily that K_4 is an M-circuit and that every 1-extension of an M-circuit is an M-circuit. This allows us to construct an infinite family of M-circuits. It is not true that every M-circuit can be obtained from a smaller M-circuit by a 1-extension, see Figure 8. Berg and Jordán showed that the latter statement becomes true, however, if we restrict our attention to 3-connected M-circuits.

Theorem 5.7 *[15, Theorem 3.8] Let $G = (V, E)$ be a 3-connected M-circuit and $x, y, z \in V$ with $xy \in E$. Then there exists a vertex $v \in V - \{x, y, z\}$ of degree three in G with non-adjacent neighbours u, w, such that $G - v + uw$ is an M-circuit.*

Figure 8 also gives an example of a 3-connected M-circuit with a vertex v of degree three such that $G - v + uw$ is not an M-circuit for all pairs of neighbours u and w of v. Thus the conclusion of Theorem 5.7 is not

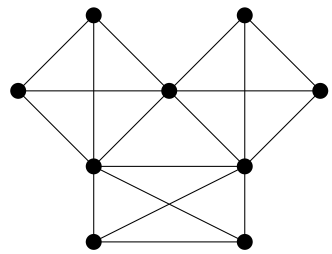

Figure 9: The above graph has exactly three M-circuits given by the three copies of K_4. It is redundantly rigid since it is rigid and each edge belongs to an M-circuit. It is not M-connected since no two edges in different K_4's belong to the same M-circuit.

valid for *all* vertices of degree three. This indicates that its proof will be more difficult than that of the corresponding result for independent graphs, Lemma 4.7(b).

5.3.2 M-connected graphs

We say that a graph G is *M-connected* if every pair of edges of G belong to an M-circuit of G. We can use the facts that M-circuits are rigid and that the union of two rigid subgraphs with at least two vertices in common is rigid, to deduce that an M-connected graph is rigid. Thus, M-connectedness is a stronger property than redundant rigidity. An example of a graph which is redundantly rigid but not M-connected is given in Figure 9.

It turns out that M-connectedness is, at least for our purpose, easier to work with than redundant rigidity. On the hand, the following lemma implies that the two concepts are equivalent for 3-connected graphs. In fact we will need a slightly stronger result. A graph G is said to be *nearly* 3-*connected* if G can be made 3-connected by the addition of at most one new edge.

Lemma 5.8 *[15] Let G be a nearly* 3*-connected graph. Then G is redundantly rigid if and only if G is M-connected.*

Proof: We have already noted that every M-connected graph is redundantly rigid. Suppose that G is redundantly rigid but not M-connected. An *M-connected component of G* is a maximal M-connected subgraph. Let

$H_1, H_2, \ldots H_q$ be the M-connected components of G. We will need the fact that $r(G) = \sum_{i=1}^{q} r(H_i)$, see [15, Section 3].

Let $X_i = V(H_i) - \cup_{j \neq i} V(H_j)$ denote the set of vertices belonging to no other M-component than H_i, and let $Y_i = V(H_i) - X_i$ for $1 \leq i \leq q$. Let $n_i = |V(H_i)|$, $x_i = |X_i|$, $y_i = |Y_i|$. Clearly, $n_i = x_i + y_i$ and $|V| = \sum_{i=1}^{q} x_i + |\cup_{i=1}^{q} Y_i|$. Moreover, we have $\sum_{i=1}^{q} y_i \geq 2|\cup_{i=1}^{q} Y_i|$. Since every edge of G is in some M-circuit, and every M-circuit has at least four vertices, we have that $n_i \geq 4$ for $1 \leq i \leq q$. Furthermore, since G is nearly 3-connected, $y_i \geq 2$ for all $1 \leq i \leq q$, and $y_i \geq 3$ for all but at most two M-components.

Since each M-connected component of G is redundantly rigid, and hence rigid, we have $r(H_i) = 2n_i - 3$ for all $1 \leq i \leq q$. Using the above inequalities we have

$$\begin{aligned} r(G) &= \sum_{i=1}^{q} r(H_i) = \sum_{i=1}^{q} (2n_i - 3) = 2\sum_{i=1}^{q} n_i - 3q \\ &\geq (2\sum_{i=1}^{q} x_i + \sum_{i=1}^{q} y_i) + \sum_{i=1}^{q} y_i - 3q \geq 2|V| + 3q - 2 - 3q = 2|V| - 2. \end{aligned}$$

This contradicts Lemma 2.3. •

5.3.3 Ear decompositions

Let $G = (V, E)$ be a graph and let $H_1, H_2, ..., H_t$ be a non-empty sequence of M-circuits of G. Let $G_j = H_1 \cup H_2 \cup ... \cup H_j$ for $1 \leq j \leq t$. We say that $H_1, H_2, ..., H_t$ is an *M-ear decomposition* of G if $G_t = G$ and, for all $2 \leq i \leq t$, we have

- $E(H_i) \cap E(G_{i-1}) \neq \emptyset \neq E(H_i) - E(G_{i-1})$, and
- no M-circuit H of G_i which satisfies

  $$E(H) \cap E(G_{i-1}) \neq \emptyset \neq E(H) - E(G_{i-1})$$

  has $E(H) - E(G_{i-1})$ properly contained in $E(H_i) - E(G_{i-1})$.

An example of an M-ear decomposition is given in Figure 10.

We will need the following result for M-connected graphs which follows from a more general result for connected matroids due to Coullard and Hellerstein [7].

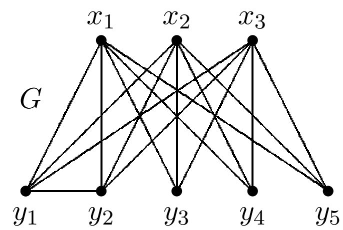

Figure 10: If $H_1 = G - y_1$, $H_2 = G - y_2$ and $H_3 = G - \{y_4, y_5\}$, then H_1, H_2, H_3 is an M-ear decomposition of G.

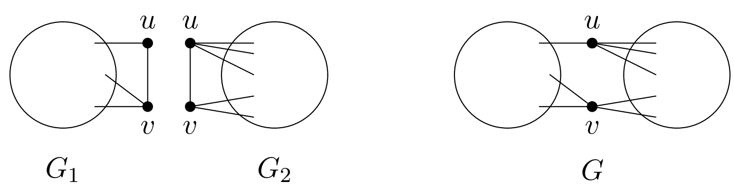

Figure 11: G is the 2-sum of G_1 and G_2

Lemma 5.9 *Let G be a graph. Then G is M-connected if and only if G has an M-ear decomposition.*

We can now state the final result which allows us to reduce 3-connected M-connected graphs to smaller M-connected graphs.

Theorem 5.10 *[15, Theorem 5.4] Let G be a 3-connected M-connected graph and $H_1, H_2, \ldots H_t$ be an M-ear decomposition of G with $t \geq 2$. Suppose that $G - e$ is not M-connected for all $e \in E(H_t) - \bigcup_{i=1}^{t-1} E(H_i)$ and for all but at most two edges $e \in E(H_t)$. Then there exists a vertex $v \in V(H_t) - \bigcup_{i=1}^{t-1} V(H_i)$ of degree three in G with non-adjacent neighbours u, w such that $G - v + uw$ is M-connected.*

We may use Theorem 5.10 to give a recursive construction for M-connected graphs. We need another new operation. If G_1 and G_2 are graphs with $V(G_1) \cap V(G_2) = \{u, v\}$ and $E(G_1) \cap E(G_2) = \{uv\}$, then we say that the graph $G = (G_1 - uv) \cup (G_2 - uv)$ is a *2-sum of G_1 and G_2*, see Figure 11.

Lemma 5.11 *[15, Lemmas 3.3,3.4] Suppose G is the 2-sum of G_1 and G_2. Then G is M-connected if and only if G_1 and G_2 are both M-connected.*

Combining Theorems 5.7 and 5.10, and Lemma 5.11, we obtain

Corollary 5.12 *[15, Corollary 5.9] Let G be a graph. Then G is M-connected if and only if G is a connected graph which can be obtained from disjoint K_4's by recursively applying edge-additions and 1-extensions to the same connected component, and 2-sums to two different connected components.*

The following special case of Corollary 5.12 when G is an M-circuit was previously obtained by Berg and Jordán by using Lemma 5.11 and Theorem 5.7.

Corollary 5.13 *[2, Theorem 4.4] Let G be a graph. Then G is an M-circuit if and only if G is a connected graph which can be obtained from disjoint K_4's by recursively applying 1-extensions to the same connected component, and 2-sums to two different connected components.*

5.3.4 Bricks

We have seen that every 3-connected M-connected graph with at least five vertices can be obtained from a smaller M-connected graph by an edge addition or a 1-extension. The last piece in our characterization of globally rigid graphs is to show that every 3-connected M-connected graph with at least five vertices can be obtained from a smaller *3-connected* M-connected graph by an edge addition or a 1-extension.

We say that a graph G is a *brick* if it is both 3-connected and M-connected.

Theorem 5.14 *[15, Theorem 6.1] Let $G = (V, E)$ be a brick with at least five vertices. Then either there exists an edge $e \in E$ such that $G - e$ is a brick, or there exists a vertex $v \in V$ of degree three in G with non-adjacent neighbours u, w such that $G - v + uw$ is a brick.*

The special case of Theorem 5.14 when G is a 3-connected M-circuit was previously obtained by Berg and Jordán [2, Theorem 5.9]. (In this special case, the minimality of an M-circuit implies that there can never exist an edge whose deletion preserves M-connectivity, so only the second reduction step in Theorem 5.14 is needed for M-circuits.)

Our proof of Theorem 5.14 is by contradiction. We suppose the theorem is false and let G be a smallest counterexample. Then $G-e$ is not a brick for all $e \in E$, i.e. either $G-e$ is not M-connected or $G-e$ is not 3-connected. If $G-e$ is not 3-connected for some edge e then we choose a 2-vertex cut $\{u,v\}$ of $G-e$ and a component F_e of $G-e-\{u,v\}$ such that F_e is as small as possible, and let H_e be the graph obtained from F_e by adding the vertices u and v, all edges of G between F_e and $\{u,v\}$, and the new edge uv.

Simialrly, $G-w+xy$ is not a brick for all $w \in V$ of degree three in G and non-adjacent neighbours x,y of w. If $G-w+xy$ is not 3-connected, we choose a 2-vertex cut $\{u,v\}$ of $G-w+xy$ and a component $F_{w,x,y}$ of $(G-w+xy)-\{u,v\}$ such that $F_{w,x,y}$ is as small as possible, and let $H_{w,x,y}$ be the graph obtained from $F_{w,x,y}$ by adding the vertices u and v, all edges of G between $F_{w,x,y}$ and $\{u,v\}$, and the new edge uv.

We now consider all such graphs H_e and $H_{w,x,y}$ and choose one, say H, with as few vertices as possible. Note that we will have at least one graph to choose from since, if no graph H_e exists then $G-e$ is 3-connected for all $e \in E$, and $G-e$ is not M-connected for all $e \in E$. Theorems 5.7 and 5.10 now imply that $G-w+xy$ is M-connected for some $w \in V$ of degree three in G and non-adjacent neighbours x,y of w. The fact that G is a counterexample now implies that $G-w+xy$ is not 3-connected and hence $H_{w,x,y}$ exists.

We complete the proof by showing that H is a brick and then applying Theorems 5.7 and 5.10 to H to find either an edge $e \in E(H)$ or a vertex $w \in V(H)$ such that $G-e$ or $G-w+xy$ contradicts the minimality of H. When we do this we seem to need the full strength of Theorems 5.7 and 5.10, i.e. we need to be able to find a vertex v avoiding the specified set of vertices in Theorem 5.7, and we need to allow H_t to contain two edges whose deletion does not destroy the M-connectedness of G in Theorem 5.10.

Theorem 5.14 and the facts that 1-extensions and edge additions preserve both M-connectedness and 3-connectedness imply the following recursive constructions for bricks and 3-connected M-circuits.

Corollary 5.15 *[15, Theorem 6.15] A graph G is a brick if and only if G can be obtained from K_4 by 1-extensions and edge additions.*

Corollary 5.16 *[2, Theorem 4] A graph G is a 3-connected M-circuit if and only if G can be obtained from K_4 by 1-extensions.*

We illustrate Corollary 5.15 by constructing the brick $K_{3,5}$ from K_4, in Figure 12.

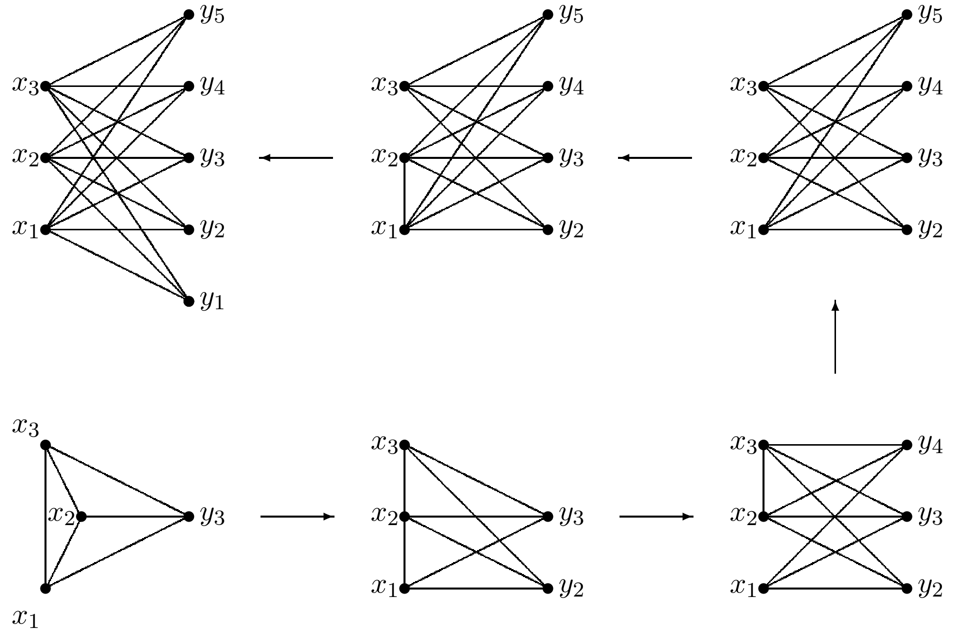

Figure 12: A construction of $K_{3,5}$ from K_4 using 1-extensions and edge additions. Since $K_{3,5} - e$ is not 3-connected for all edges e, the first and last operations used in the construction must be 1-extensions. Since $K_{3,5}$ is not an M-circuit, at least one operation in the construction must be an edge addition. This shows that one may need to alternate between the two operations of Corollary 5.15 while building up a brick from K_4.

5.4 The final goal

We can now obtain our promised characterization of global rigidity in $\mathbb{R}^2$.

Theorem 5.17 *[15, Theorem 7.1] Let (G, p) be a 2-dimensional generic framework. Then (G, p) is globally rigid if and only if either G is a complete graph on at most three vertices or G is 3-connected and redundantly rigid.*

Proof: Necessity follows from Lemma 5.1 and Theorem 5.2. Sufficiency follows from Theorem 5.5 and Corollary 5.15. •

The special case of Theorem 5.17 when G is a 3-connected M-circuit was obtained previously by Berg and Jordán [2, Theorem 6.1].

It follows from Theorem 5.17 that the global rigidity of 2-dimensional frameworks is a generic property. We say that a graph G is *globally rigid in* $\mathbb{R}^2$ if every generic realization of G in $\mathbb{R}^2$ is globally rigid. We can use Theorems 4.12 and 5.17 to deduce that sufficiently highly connected graphs are globally rigid in $\mathbb{R}^2$.

Corollary 5.18 *[15, Theorem 7.2] Let G be a 6-connected graph. Then $G - \{e_1, e_2\}$ is globally rigid in $\mathbb{R}^2$ for all edges e_1, e_2 of G.*

6 Further Reading

More information on the rigidity of graphs and frameworks can be found in the survey article by Whiteley [22], and the books by Graver [11] and Graver, Servatius and Servatius [12]. Some partial results, conjectures, and additional references on the rigidity of graphs in $\mathbb{R}^3$ can be found in [16].

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References

[1] L. Asimow and B. Roth, Rigidity of graphs II, *J. Math. Anal. Appl.* 68 (1979) 171-190.

[2] A. Berg and T. Jordán, A proof of Connelly's conjecture on 3-connected circuits of the rigidity matroid, *J. Combinatorial Theory, Ser. B.*, Vol. 88, 77-97, 2003.

[3] A. L. Cauchy, Sur les polygons et les polyédres, *Second Memoire, J. École Polytechnique* 19 (1813) 87-98.

[4] R. Connelly, A flexible sphere, *Math Intelligencer* 1 (1982) 130-131.

[5] R. Connelly, On generic global rigidity, in *Applied Geometry and Discrete Mathematics*, 147–155, DIMACS Ser. Discrete Math. Theoret. Comput. Sci., 4, Amer. Math. Soc., Providence, RI, 1991.

[6] R. Connelly, Generic global rigidity, *Discrete Comput. Geom.*, 33 (2005) 549-563.

[7] C.R. Coullard and L. Hellerstein, Independence and port oracles for matroids, with an application to computational learning theory, *Combinatorica* 16 (1996), no. 2, 189–208.

[8] L. Euler, *Opera Postuma*, vol. 1. Petropoli, 1862, 494-496. Euler Archive index number E819, http://math.dartmouth.edu/ euler/

[9] H. Gluck, Almost all simply connected closed surfaces are rigid, in *Geometric topology*, Lecture Notes in Math., Vol. 438, Springer, Berlin, 1975, 225–239.

[10] S. J. Gortler, A. D. Healy and D. P. Thurston, Characterizing global rigidity, preprint, http://arxiv.org/abs/0710.0926

[11] J. Graver, *Counting on Frameworks*, Mathematical Association of America, Dolciani Mathematical Expositions number 25, 2001.

[12] J. Graver, B. Servatius, and H. Servatius, *Combinatorial Rigidity*, AMS Graduate Studies in Mathematics Vol. 2, 1993.

[13] B. Hendrickson, Conditions for unique graph realizations, *SIAM J. Comput.* **21** (1992), no. 1, 65-84.

[14] L. Henneberg, *Die graphische Statik der starren Systeme*, Leipzig 1911.

[15] B. Jackson and T. Jordán, Connected rigidity matroids and unique realizations of graphs, *J. Combinatorial Theory Ser. B*, 94 (2005) 1-29.

[16] B. Jackson and T. Jordán, On the rank function of the 3-dimensional rigidity matroid, *Int. J. Computational Geometry and Applications*, 16 (2006) 415-429.

[17] B. Jackson, T. Jordán and Z. Szabadka, Globally Linked Pairs of Vertices in Equivalent Realizations of Graphs, *Discrete and Computational Geometry* 35 (2006) 493-512.

[18] G. Laman, On graphs and rigidity of plane skeletal structures, *J. Engineering Math.* 4 (1970), 331-340.

[19] L. Lovász and Y. Yemini, On generic rigidity in the plane, *SIAM J. Algebraic Discrete Methods* 3 (1982), no. 1, 91–98.

[20] J.W. Milnor, *Topology from the differentiable viewpoint*, University Press of Virginia, Charlottesville (1965).

[21] J.B. Saxe, Embeddability of weighted graphs in k-space is strongly NP-hard, Tech. Report, Computer Science Department, Carnegie-Mellon University, Pittsburgh, PA, 1979.

[22] W. Whiteley, Some matroids from discrete applied geometry, in *Matroid theory*, Contemp. Math., 197, Amer. Math. Soc., Providence, RI, 1996, 171–311.

A Appendix

We give a simple proof of a strengthening of Theorem 3.8 which was obtained by several participants at the Levico meeting, including Pierre Charbit, Daniel Dadush, Támas Király, and Adam Watson.

We say that a set $S \subseteq \mathbb{R}$ is $\{0, \pm 1, \pm 2\}$-*sum-free* if the only solution to the equation $\sum_{x \in S} \alpha_x x$ with $\alpha_x \in \{0, \pm 1, \pm 2\}$ for all $x \in S$ is the trivial solution $\alpha_x = 0$ for all $x \in S$.

Theorem A.1 *Let (G, p_1) be a 1-dimensional framework. Suppose that $G = (V, E)$ is 2-connected and $p_1(V)$ is $\{0, \pm 1, \pm 2\}$-sum-free. Then (G, p_1) is globally rigid.*

Proof: Let (G, p_2) be a 1-dimensional framework which is equivalent to (G, p_1) and let $x, y \in V$. It suffices to show that $|p_1(x) - p_1(y)| = |p_2(x) - p_2(y)|$. Let $C = v_1 v_2 \dots v_m v_1$ be a cycle in G which contains x and y. For each $i \in \{1, 2\}$, let $F_i^+ = \{v_j v_{j+1} \in E(C) \;:\; p_i(v_j) < p_i(v_{j+1})\}$ and $F_i^- = \{v_j v_{j+1} \in E(C) \;:\; p_i(v_j) > p_i(v_{j+1})\}$. Then, for each $i \in \{1, 2\}$, we have

$$\sum_{e \in F_i^+} \|e\| = \sum_{e \in F_i^-} \|e\|. \tag{6}$$

If $F_1^+ = F_2^+$ or $F_1^+ = F_2^-$ then (C, p_1) and (C, p_2) are congruent. This implies that $|p_1(x) - p_1(y)| = |p_2(x) - p_2(y)|$. Thus we may assume that $F_2^- \neq F_1^+ \neq F_2^+$. We may rewrite (6) as two simultaneous equaltions:

$$\sum_{e \in F_1^+ \cap F_2^+} \|e\| + \sum_{e \in F_1^+ - F_2^+} \|e\| = \sum_{e \in F_1^- \cap F_2^-} \|e\| + \sum_{e \in F_1^- - F_2^-} \|e\| \tag{7}$$

$$\sum_{e \in F_2^+ \cap F_1^+} \|e\| + \sum_{e \in F_2^+ - F_1^+} \|e\| = \sum_{e \in F_2^- \cap F_1^-} \|e\| + \sum_{e \in F_2^- - F_1^-} \|e\|. \tag{8}$$

Subtracting (8) from (7) we obtain

$$\sum_{e \in F_1^+ - F_2^+} \|e\| - \sum_{e \in F_2^+ - F_1^+} \|e\| = \sum_{e \in F_1^- - F_2^-} \|e\| - \sum_{e \in F_2^- - F_1^-} \|e\|.$$

Since $F_1^+ - F_2^+ = F_2^- - F_1^-$ and $F_1^- - F_2^- = F_2^+ - F_1^+$, we have

$$\sum_{e \in F_1^+ - F_2^+} \|e\| - \sum_{e \in F_1^- - F_2^-} \|e\| = 0. \tag{9}$$

Since $(F_1^+ - F_2) \cup (F_1^- - F_2^-)$ is a proper subset of $E(C)$, some vertex v of C is incicent with exactly one edge of $(F_1^+ - F_2) \cup (F_1^- - F_2^-)$. This implies that $p(v)$ will have coefficient ± 1 when (9) is expanded as an equality involving $p(v_i)$, $v_i \in V(C)$, which contradicts the hypothesis that $p_1(V)$ is $\{0, \pm 1, \pm 2\}$-sum-free. •

B Appendix

We give a result obtained by Jim Geelan during the Levico meeting which characterizes the graphs G which can be realized as a 1-dimensional framework which is not globally rigid.

Theorem B.1 *Let $G = (V, E)$ be a graph. Then G can be realized as a a 1-dimensional framework which is not globally rigid if and only if G is a spanning subgraph of the line graph of a bipartite graph which has at least two vertices on each side of its bipartition.*

Proof: We first show sufficiency. Let $H = (X \cup Y, E)$ be a bipartite graph with $X = \{x_0, x_1 \dots x_s\}$, $Y = \{y_0, y_1 \dots y_t\}$, and $s, t \geq 1$. It suffices to show that if G is the line graph of H, then G can be realized as a 1-dimensional framework which is not globally rigid. Consider first the realization (G, p) in the 2-dimensional $s \times t$ grid where an edge $x_i y_j$ of H is mapped onto the point (i, j) in the grid. Note that all edges of G are either vertical or horizontal in (G, p). Let (G, p_1) be obtained from (G, p) by rotating each point $p(x_i y_j)$ by ninety degrees clockwise about the point $(i, 0)$, for all $0 \leq i \leq s$. Similarly, let (G, p_2) be obtained from (G, p) by rotating each point $p(x_i y_j)$ by ninety degrees anticlockwise about the point $(i, 0)$. Then (G, p_1) and (G, p_2) are equivalent but not congruent 1-dimensional frameworks.

We next show necessity. Suppose that (G, p_1) and (G, p_2) are equivalent but not congruent 1-dimensional frameworks. Let

$$E^* = \{uv \ : \ u, v \in V \text{ and } |p_1(u) - p_1(v)| = |p_2(u) - p_2(v)|\}.$$

Since (G, p_1) and (G, p_2) are equivalent, $E \subseteq E^*$. Hence it will suffice to show that $G^* = (V, E^*)$ is the line graph of a bipartite graph which has at least two vertices on each side of its bipartition. Let

$$E^+ = \{uv \ : \ u, v \in V \text{ and } p_1(u) - p_1(v) = p_2(u) - p_2(v)\},$$

$$E^- = \{uv \ : \ u, v \in V \text{ and } p_1(u) - p_1(v) = p_2(v) - p_2(u)\},$$

and put $G^+ = (V, E^+)$ and $G^- = (V, E^-)$. We shall show that G^+ and G^- can both be expressed as the disjoint union of complete graphs. Suppose $uv, vw \in E^+$. Then $p_1(u) - p_1(w) = [p_1(u) - p_1(v)] + [p_1(v) - p_1(w)] = [p_2(u) - p_2(v)] + [p_2(v) - p_2(w)] = p_2(u) - p_2(w)$. Thus G^+ is equal to its own transitive closure and hence is a disjoint union of complete graphs. A similar proof holds for G^-.

Let $X = \{G_1^+, G_1^+, \ldots, G_s^+\}$ and $Y = \{G_1^-, G_1^-, \ldots, G_t^-\}$ be the sets of maximal complete subgraphs of G^+ and G^-, respectively. Let H be the bipartite graph with bipartition (X, Y) in which G_i^+ is adjacent to G_j^- if and only if G_i^+ and G_j^- have a non-empty intersection in G^*. Then G^* is the line graph of H. Furthermore, if $s = 1$ or $t = 1$, then G^* would be a complete graph. This would contradict the assumption that (G, p_1) and (G, p_2) are not congruent. Thus $s \geq 2$ and $t \geq 2$.