## Introduction to Probability - 2010/11

These notes are a summary of what was lectured. They do not contain all examples or explanations and are NOT a substitute for your own notes.

## 1 Sample Space and Events

An event is random if we cannot predict it with certainty. This could be because it happens "by chance" (eg tossing a coin) or because its causes are beyond our knowledge (e.g. revealing the top card of a previously shuffled deck). Probability is concerned with quantifying randomness. It is useful and mathematically appealing.

The general setting is we perform an experiment and record an outcome. Outcomes must be precisely specified, mutually exclusive and cover all possibilities.

Definition The sample space of the experiment is the set of all possible outcomes for the experiment. It will usually be denoted by $S$. An event is a subset of the sample space. The event occurs if the actual outcome is an element of the set. An event comprising of just one element of $S$ is called simple or elementary.
Example A coin is tossed three times and the sequence of heads/tails is recorded.

$$
\mathcal{S}=\{h h h, h h t, h t h, h t t, t h h, t h t, t t h, t t t\}
$$

where, for example, $h t t$ means the first toss is a head, the second is a tail, the third is a tail. If $E$ is the event "the second toss is a tail" then

$$
E=\{h t h, h t t, t t h, t t t\} .
$$

There are lots more examples in your lecture notes.
We have used the language and notation of sets here. To proceed we will need to introduce some more notation, definitions and basic theory of sets and functions. This is the content of the next two sections. This material is part of the fundamental language in which mathematics is done. It is important, for this course and others, that you become comfortable with it.

## 2 Sets

A set is a collection of distinct objects (order and repetitions are irrelevant).
A set can be specified in various ways.

- By listing the objects in it between braces $(\{\}$,$) separated by commas.$ eg $\{1,2,3,4\}$.
- By giving a rule eg $\{x: x$ is an even integer $\}$. (We read this as "the set of all $x$ such that $x$ is an even integer". The braces tell us it is a set and the colon stands for 'such that'.)
- When a set has too many elements to list we should define it using a rule. Alternatively we can define it by just listing enough elements to determine the rule. eg $\{2,4,6,8, \ldots\}$ is the set of positive even integers and $\{2,4,6,8, \ldots 1000\}$ is the set of even integers between 2 and 1000 .

If $X$ is a set we write $x \in X$ to mean that the object $x$ is in the set $X$ and say that $x$ is an element of $X$. If $x$ is not an element of $X$ then we write $x \notin X$.

Two sets are equal if they contain precisely the same set of elements.
Definition The empty set is the set with no elements; it is denoted by $\emptyset$.
Definition If $X$ is finite then the cardinality of $X$ is the number of elements of $X$; it is denoted by $|X|$.
Definition If every element of $X$ is also an element of $Y$ then we say that $X$ is a subset of $Y$ and write $X \subseteq Y$ (or $X \subset Y$ ).

### 2.1 Operations on Sets

Let $X$ and $Y$ be sets.

- $X \cup Y$ ("X union $Y$ ") is the set of elements of $X$ or $Y$ (or both)
- $X \cap Y$ (" $X$ intersection $Y$ ") is the set of elements of both $X$ and $Y$
- $X \backslash Y$ is the set of elements in $X$ but not in $Y$
- $X \triangle Y$ ("symmetric difference of $X$ and $Y$ ") is the set of elements in either $X$ or $Y$ but not both
- If we are considering sets which are subsets of some fixed set $S$ and $X \subseteq S$ then $X^{c}$ ("the complement of $X$ in $S$ ") is $S \backslash X$ (the set of all elements of $S$ which are not elements of $X$ ).

If $X, Y$ are events in a sample space $S$ then the set we get by combining $X$ and $Y$ with one of the above operations can also be interpretated as an event:

- $X \cup Y$ is the event "either $X$ or $Y$ occurs"
- $X \cap Y$ is the event "both $X$ and $Y$ occur"
- $X \backslash Y$ is the event " $X$ occurs but $Y$ doesn't"
- $X \triangle Y$ is the event "exactly one of $X$ and $Y$ occurs"
- $X^{c}$ is the event " $X$ doesn't occur"

Definition The sets $X$ and $Y$ are disjoint if $X \cap Y=\emptyset$.

### 2.2 Set Identities

Lemma 2.1. Suppose $X, Y$ and $Z$ are sets. Then
a) Commutative laws
i) $X \cap Y=Y \cap X$
ii) $X \cup Y=Y \cup X$
iii) $X \triangle Y=Y \triangle X$
b) Associative laws
i) $X \cup(Y \cup Z)=(X \cup Y) \cup Z$
ii) $X \cap(Y \cap Z)=(X \cap Y) \cap Z$
c) Distributive laws
i) $X \cap(Y \cup Z)=(X \cap Y) \cup(X \cap Z)$
ii) $X \cup(Y \cap Z)=(X \cup Y) \cap(X \cup Z)$
d) De Morgan laws
i) $(X \cap Y)^{c}=X^{c} \cup Y^{c}$
ii) $(X \cup Y)^{c}=X^{c} \cap Y^{c}$

Each of the above identities asserts that two sets are equal. The general strategy for proving such an identity, $A=B$ say, is as follows.

- First Step. Prove that if $x \in A$ then $x \in B$. This shows that $A \subseteq B$.
- Second Step. Prove that and if $x \in B$ then $x \in A$. This shows that $B \subseteq A$.
I will give a proof of Lemma 2.1(d)(i) as an example of this strategy. You should check the other parts of the Lemma yourself by constructing Venn diagrams and formal proofs.
Proof of Lemma 2.1(d)(i) First Step. Suppose $z \in(X \cap Y)^{c}$. Then $z \notin(X \cap Y)$. This implies that either $z \notin X$ or $z \notin Y$. Hence $z \in X^{c} \cup Y^{c}$. Since this holds for all $z \in(X \cap Y)^{c}$ we have $(X \cap Y)^{c} \subseteq X^{c} \cup Y^{c}$.

Second Step. Suppose $z \in X^{c} \cup Y^{c}$. Then either $z \in X^{c}$ or $z \in Y^{c}$. So either $z \notin X$ or $z \notin Y$. This implies that $z \notin X \cap Y$. Hence $z \in(X \cap Y)^{c}$. Since this holds for all $z \in X^{c} \cup Y^{c}$ we have $X^{c} \cup Y^{c} \subseteq(X \cap Y)^{c}$.

Thus $(X \cap Y)^{c}=X^{c} \cup Y^{c}$.

### 2.3 Ordered Pairs and Cartesian Products

We denote the ordered pair " $x$ then $y$ " by $(x, y)$. So $(1,2) \neq(2,1)$ unlike for sets where $\{1,2\}=\{2,1\}$.
Definition The Cartesian product of two sets $A$ and $B$, denoted by $A \times B$, is defined by

$$
A \times B=\{(a, b): a \in A, b \in B\}
$$

We denote $A \times A$ by $A^{2}$.
More generally, an ordered $n$-tuple is written $\left(x_{1}, x_{2}, \ldots, x_{n}\right)$ and the Cartesian product of $n$ sets $A_{1}, A_{2}, \ldots, A_{n}$ is defined as

$$
A_{1} \times A_{2} \times \cdots \times A_{n}=\left\{\left(a_{1}, a_{2}, \ldots, a_{n}\right): a_{i} \in A_{i} \text { for } 1 \leq i \leq n\right\} .
$$

We write $A^{n}$ for the Cartesian product $\underbrace{A \times A \times \cdots \times A}_{n \text { terms }}$.

