MTH4107 Introduction to Probability – 2010/11

Exercise Sheet 9

These questions are designed to help you understand joint distributions. You should discuss them in your week 11 exercise class. It is important that you make a serious attempt to do questions Q1-Q3 before week 12 lectures begin. Questions AQ1-AQ2 are for additional practice. You should attempt them when you have time.
In addition to your lecture notes material relating to joint distributions can be found in Devore, Chapter 5 Sections 5.1-2. or Ross, Chapter 6 Sections 6.1-2, 6.4.

Q1. Two fair standard dice are rolled. Let A be the number of 1s seen and B be the number of 2s seen in the outcome.

(a) Find the joint probability mass function of A and B.

(b) Determine whether A and B are independent.

(c) Find the covariance and the correlation coefficient of A and B.

Q2. Suppose that X, Y, Z are random variables with $X \sim \text{Bin}(7, 1/6)$, $Y \sim \text{Geom}(1/2)$, and $Z \sim \text{Poisson}(6)$. Suppose further that X and Y are independent but that X and Z are not independent. State which of the following values can be determined from this information, and find the ones which can be determined: (i) $\mathbb{E}(X+Y)$; (ii) $\mathbb{E}(X+Z)$; (iii) $\mathbb{E}(X+2Y+3Z)$; (iv) $\text{Var}(X+Y)$; (v) $\text{Var}(X+Z)$; (vi) $\text{Var}(X+2Y+3Z)$.

Q3. Let $X_1, X_2, \ldots, X_n$ be discrete random variables defined on the same sample space and $c_1, c_2, \ldots, c_n \in \mathbb{R}$ be constants.
(a) Use Propositions 11.2(a) and 13.2, and induction on n to show that

$$\mathbb{E}\left(c_1X_1 + c_2X_2 + \ldots + c_nX_n\right) = c_1\mathbb{E}(X_1) + c_2\mathbb{E}(X_2) + \ldots + c_n\mathbb{E}(X_n)$$

(b) Suppose $X_1, X_2, \ldots, X_n$ are independent. Use Propositions 11.3(b) and 13.4, and induction on n to show that

$$\text{Var}\left(c_1X_1 + c_2X_2 + \ldots + c_nX_n\right) = c_1^2\text{Var}(X_1) + c_2^2\text{Var}(X_2) + \ldots + c_n^2\text{Var}(X_n)$$

AQ1. Let X be the number of fish caught by a fisherman and Y be the number of fish caught by a second fisherman in one afternoon of fishing. Suppose that X is distributed Poisson(λ) and Y is distributed Poisson(μ). Suppose further that X and Y are independent random variables.
(a) Show that

$$\mathbb{P}(X+Y=n) = \sum_{k=0}^{n} e^{-\lambda}\frac{\lambda^k}{k!}e^{-\mu}\frac{\mu^{n-k}}{(n-k)!}.$$

(b) Use the Binomial Theorem to deduce that

$$\mathbb{P}(X+Y=n) = e^{-(\lambda+\mu)}\frac{\lambda^n}{n!}$$

and hence that $X+Y \sim \text{Poisson}(\lambda+\mu)$.

AQ2 Suppose that a bag contains B balls, R of which are red and the remaining $B-R$ are white. We make a selection of n balls without replacement. Let X be the number of red balls among the n balls chosen. Recall that the random variable X has the *hypergeometric distribution* (see Section 12.3 in the notes). This question leads you through the calculation of the expectation and variance of X. The method is similar to the way we used joint distributions to calculate the expectation and variance of a binomial random variable.

(a) Regard the selection as being made with order and define random variables $X_1, \ldots, X_n$ by

$$X_i = \begin{cases} 0 & \text{if the } i\text{th ball chosen is blue} \\ 1 & \text{if the } i\text{th ball chosen is red} \end{cases}$$

(b) Find the probability mass function of each X_i and deduce that $\mathbb{E}(X_i) = \frac{R}{B}$. (*Hint: Each X_i has the same probability distribution.*)

(c) Express X in terms of the X_i.

(d) Hence show that the expectation of X is $\frac{nR}{B}$.

(e) Determine the probability mass function of X_i^2 and deduce that $\mathbb{E}(X_i^2) = \frac{R}{B}$.

(f) Determine the probability mass function of X_iX_j when $i \neq j$ and deduce that $\mathbb{E}(X_iX_j) = \frac{R(R-1)}{B(B-1)}$.

(g) Use (c), (e) and (f) to show that

$$\mathbb{E}(X^2) = n\frac{R}{B} + n(n-1)\frac{R(R-1)}{B(B-1)}.$$

(h) Deduce that

$$\text{Var}(X) = n\frac{R}{B}\left(1 - \frac{R}{B}\right)\frac{B-n}{B-1}.$$