## MTH4107 Introduction to Probability - 2010/11

## Solutions to Exercise Sheet 8

Q1. This question uses the basic properties of expectation and variance that we proved in Propositions 11.2 and 11.3. Make sure you can understand which parts of those results are being used at each step. (I've told you for the first two).
(a) $\mathbb{E}(3 X)=3 \mathbb{E}(X)=15$ (using part (a) of Proposition 11.2)
(b) $\operatorname{Var}(3 X)=3^{2} \operatorname{Var}(X)=9 \times 2 / 3=6$ (using part (a) of Proposition 11.3)
(c) $\mathbb{E}(4-3 X)=\mathbb{E}(4)+\mathbb{E}(-3 X)=4-3 \mathbb{E}(X)=4-15=-11$
(d) $\operatorname{Var}(4-3 X)=\operatorname{Var}(-3 X)=(-3)^{2} \operatorname{Var}(X)=9 \times 2 / 3=6$
(e) $\mathbb{E}\left(4-3 X^{2}\right)=4-3 \mathbb{E}\left(X^{2}\right)$ but what is $\mathbb{E}\left(X^{2}\right)$ ?

By the definition of variance $\operatorname{Var}(X)=\mathbb{E}\left(X^{2}\right)-(\mathbb{E}(X))^{2}$ so $\mathbb{E}\left(X^{2}\right)=2 / 3+5^{2}=$ 77/3. Hence,

$$
\mathbb{E}\left(4-3 X^{2}\right)=4-3 \times 77 / 3=-73
$$

(Hopefully nobody was tempted to write $\mathbb{E}\left(X^{2}\right)=5^{2}$.)

Q2. There are several equally correct answers to this question. The key point is that the binomial distribution is used for the number of successes in a fixed number of trials while the geometric distribution is used for the number of trials up to and including the first success.
(a) The number of 1 s seen in the first 6 rolls has the $\operatorname{Bin}(6,1 / 6)$ distribution. Its expectation is $6 \times 1 / 6=1$ and its variance is $6 \times 1 / 6 \times 5 / 6=5 / 6$
(b) The number of odd numbers seen in the first 8 rolls has the $\operatorname{Bin}(8,1 / 2)$ distribution. Its expectation is $8 \times 1 / 2=4$ and its variance is $8 \times 1 / 2 \times 1 / 2=2$
(c) The number of rolls needed to get a 6 has the $\operatorname{Geom}(1 / 6)$ distribution. Its expectation is $\frac{1}{1 / 6}=6$ and its variance is $\frac{5 / 6}{(1 / 6)^{2}}=30$.
(d) The number of rolls needed to get a number bigger than 2 has the Geom(2/3) distribution. Its expectation is $\frac{1}{2 / 3}=3 / 2$ and its variance is $\frac{1 / 3}{(2 / 3)^{2}}=3 / 4$

Q3.
(a) $\mathbb{P}(A=3)=\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}=\frac{4}{27}$.
(b) $\mathbb{P}(A \leq 3)=1-\mathbb{P}(A>3)=1-\left(\frac{2}{3}\right)^{3}=\frac{19}{27}$.

You could equally well say

$$
\mathbb{P}(A \leq 3)=\mathbb{P}(A=1)+\mathbb{P}(A=2)+\mathbb{P}(A=3)=\frac{1}{3}+\frac{2}{3} \times \frac{1}{3}+\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3}=\frac{19}{27}
$$

(c) $\mathbb{P}(B=2)=e^{-3 \frac{3^{2}}{2!}}=e^{-3 \frac{9}{2}}=\frac{9}{2 e^{3}}$
(d)

$$
\begin{aligned}
\mathbb{P}(B>2) & =1-\mathbb{P}(B=0)-\mathbb{P}(B=1)-\mathbb{P}(B=2) \\
& =1-e^{-3} \frac{3^{0}}{0!}-e^{-3} \frac{3^{1}}{1!}-e^{-3} \frac{3^{2}}{2!} \\
& =1-e^{-3}-3 e^{-3}-e^{-3} \frac{9}{2} \\
& =1-\frac{17}{2 e^{3}} .
\end{aligned}
$$

Q4 We have

$$
\begin{equation*}
\mathbb{E}\left(X^{2}\right)=\sum_{k=1}^{\infty} k^{2} p(1-p)^{k-1}=\sum_{r=0}^{\infty}(r+1)^{2} p(1-p)^{r} \tag{1}
\end{equation*}
$$

where the second equality follows by putting $r=k-1$. We also have

$$
\begin{equation*}
(1-p) \mathbb{E}\left(X^{2}\right)=\sum_{k=1}^{\infty} k^{2} p(1-p)^{k}=\sum_{r=0}^{\infty} r^{2} p(1-p)^{r} \tag{2}
\end{equation*}
$$

where the second equality follows by putting $r=k$ and using the fact that the $r=0$ term in the last summation is zero. Subtracting (2) from (1) gives

$$
\begin{equation*}
[1-(1-p)] \mathbb{E}\left(X^{2}\right)=\sum_{r=0}^{\infty}(2 r+1) p(1-p)^{r}=2 \sum_{r=0}^{\infty} r p(1-p)^{r}+p \sum_{r=0}^{\infty}(1-p)^{r} \tag{3}
\end{equation*}
$$

Now $[1-(1-p)] \mathbb{E}\left(X^{2}\right)=p \mathbb{E}\left(X^{2}\right)$,

$$
\sum_{r=0}^{\infty} r p(1-p)^{r}=(1-p) \sum_{r=1}^{\infty} r p(1-p)^{r-1}=(1-p) E(X)
$$

and

$$
\sum_{r=0}^{\infty}(1-p)^{r}=\frac{1}{1-(1-p)}=\frac{1}{p}
$$

by the formula for the sum of a geometric progression. Substituting into (3) and using the fact that $\mathbb{E}(X)=1 / p$ gives $p \mathbb{E}\left(X^{2}\right)=\frac{2(1-p)}{p}+1$, so

$$
\mathbb{E}\left(X^{2}\right)=\frac{2(1-p)}{p^{2}}+\frac{1}{p}
$$

Hence

$$
\operatorname{Var}(X)=\mathbb{E}\left(X^{2}\right)-\mathbb{E}(X)^{2}=\frac{2(1-p)}{p^{2}}+\frac{1}{p}-\frac{1}{p^{2}}=\frac{1-p}{p^{2}}
$$

Q5 (a) We have

$$
\begin{aligned}
\mathbb{E}(X[X-1]) & =\sum_{k \in \operatorname{Range}(X)} k(k-1) \mathbb{P}(X=k) \\
& =\sum_{k=0}^{n} k(k-1)\binom{n}{k} p^{k}(1-p)^{n-k} \\
& =\sum_{k=2}^{n} k(k-1) \frac{n!}{k!(n-k)!} p^{k}(1-p)^{n-k} \\
& =n(n-1) p^{2} \sum_{k=2}^{n} \frac{(n-2)!}{(k-2)!(n-k)!} p^{k-2}(1-p)^{n-k}
\end{aligned}
$$

where the third equality uses the fact that the $k=0$ and $k=1$ term in the summation are zero. We can now put $m=n-2$ and $i=k-2$ in the last equality to obtain

$$
\begin{aligned}
\mathbb{E}(X[X-1]) & =n(n-1) p^{2} \sum_{i=0}^{m} \frac{m!}{i!(m-i)!} p^{i}(1-p)^{m-i} \\
& =n(n-1) p^{2} \sum_{i=0}^{m}\binom{m}{i} p^{i}(1-p)^{m-i} \\
& =n(n-1) p^{2}[p+(1-p)]^{m} \\
& =n(n-1) p^{2}
\end{aligned}
$$

where the third equality uses the Binomial Theorem.
Since $\mathbb{E}(X[X-1])=\mathbb{E}\left(X^{2}-X\right)=\mathbb{E}\left(X^{2}\right)-\mathbb{E}(X)$ we have $\mathbb{E}\left(X^{2}\right)=\mathbb{E}(X[X-1])+$ $\mathbb{E}(X)$. Since $\mathbb{E}(X)=n p$ this gives
$\operatorname{Var}(X)=\mathbb{E}\left(X^{2}\right)-\mathbb{E}(X)^{2}=\mathbb{E}(X[X-1])+\mathbb{E}(X)-\mathbb{E}(X)^{2}=n(n-1) p^{2}+n p-n^{2} p^{2}=n p(1-p)$
(b) We have

$$
\begin{aligned}
\mathbb{E}(X[X-1]) & =\sum_{k \in \text { Range }(X)} k(k-1) \mathbb{P}(X=k) \\
& =\sum_{k=0}^{\infty} k(k-1) e^{-\lambda} \frac{\lambda^{k}}{k!} \\
& =e^{-\lambda} \sum_{k=2}^{\infty} k(k-1) \frac{\lambda^{k}}{k!} \\
& =\lambda^{2} e^{-\lambda} \sum_{k=2}^{\infty} \frac{\lambda^{k-2}}{(k-2)!} \\
& =\lambda^{2} e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!} \\
& =\lambda^{2}
\end{aligned}
$$

where:

- the third equality uses the fact that the $k=0$ and $k=1$ terms in the summation are zero;
- the fifth equality uses the substitution $i=k-2$;
- the last equality uses the Tayler expansion $e^{\lambda}=\sum_{i=0}^{\infty} \frac{\lambda^{i}}{i!}$ for any $\lambda \in \mathbb{R}$.

Since $\mathbb{E}(X[X-1])=\mathbb{E}\left(X^{2}-X\right)=\mathbb{E}\left(X^{2}\right)-\mathbb{E}(X)$ we have $\mathbb{E}\left(X^{2}\right)=\mathbb{E}(X[X-1])+$ $\mathbb{E}(X)$. Since $\mathbb{E}(X)=\lambda$ this gives

$$
\operatorname{Var}(X)=\mathbb{E}\left(X^{2}\right)-\mathbb{E}(X)^{2}=\mathbb{E}(X[X-1])+\mathbb{E}(X)-\mathbb{E}(X)^{2}=\lambda^{2}+\lambda-\lambda^{2}=\lambda
$$

AQ1 The Range of $N$ is $\{0,1,2,3\}$ and the probability mass function is

$$
\begin{array}{r|cccc}
n & 0 & 1 & 2 & 3 \\
\hline P(N=n) & 1 / 2 & 5 / 16 & 1 / 8 & 1 / 16
\end{array}
$$

Please let me know if you have any comments or corrections

