MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 5

Q1.

(a) If we record the outcome by listing the children in the order they were born, writing b for boy and g for girl, then the sample space is

$$\{gg, gbg, gbbg, gbbb, bgg, bgbg, bgbb, bbgg, bbgb, bbbg, bbbb\}.$$

(b) By independence and the fact that each child is equally likely to be a boy or girl we have for example

$$\mathbb{P}(gg) = \left(\frac{1}{2}\right)^2 = \frac{1}{4}.$$

Similarly,

$$\mathbb{P}(gbg) = \left(\frac{1}{2}\right)^3 = \frac{1}{8}.$$

So $\mathbb{P}(gg) \neq \mathbb{P}(gbg)$ and so not all elements are equally likely.

Note that to answer this part we don't need to calculate the probability of *every* element of the sample space. It suffices just to find two which have different probabilities. (Of course, you could have chosen a different pair from me).

(c)

$$\mathbb{P}(2 \text{ girls}) = \mathbb{P}(\{gg, gbg, gbbg, bgg, bgbg, bbgg\}) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{8} + \frac{1}{16} + \frac{1}{16} = \frac{11}{16}.$$

Here we have used the same method as part (b) to calculate the probability of each of the outcomes making up the event.

(d) Let A be the event "they have two girls" and B be the event "their first child is a boy". We want $\mathbb{P}(A|B)$. By definition

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

We have $\mathbb{P}(B) = 1/2$ and we can find $\mathbb{P}(A \cap B)$ in a similar way to (c):

$$\mathbb{P}(A \cap B) = \mathbb{P}(\{bgg, bgbg, bbgg\}) = \frac{1}{8} + \frac{2}{16} = \frac{1}{4}.$$

So

$$\mathbb{P}(A|B) = \frac{1/4}{1/2} = \frac{1}{2}.$$

Q2. We have $|S| = 36$, $|E| = |O| = 18$, $|Q| = |\{1, 4, 9, 16, 25, 26\}| = 6$, $|E \cap O| = 0$, $|Q \cap E| = |\{4, 16, 36\}| = 3$, $|Q \cap O| = |\{1, 9, 25\}| = 3$. All choices are equally likely so the probability of any event X is $\frac{|X|}{|S|}$.

(a) $\mathbb{P}(E) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(O) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(E \cap O) = \frac{0}{36} = 0$. It follows that

$$\mathbb{P}(E)\mathbb{P}(O) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \neq 0 = \mathbb{P}(E \cap O)$$

and so E and O are not independent.

(b) $\mathbb{P}(E) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(Q) = \frac{6}{36} = \frac{1}{6}$, $\mathbb{P}(E \cap Q) = \frac{3}{36} = \frac{1}{12}$. It follows that

$$\mathbb{P}(E)\mathbb{P}(Q) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} = \mathbb{P}(E \cap Q)$$

and so E and Q are independent.

(c) $\mathbb{P}(O) = \frac{18}{36} = \frac{1}{2}$, $\mathbb{P}(Q) = \frac{6}{36} = \frac{1}{6}$, $\mathbb{P}(O \cap Q) = \frac{3}{36} = \frac{1}{12}$. It follows that

$$\mathbb{P}(O)\mathbb{P}(Q) = \frac{1}{2} \times \frac{1}{6} = \frac{1}{12} = \mathbb{P}(O \cap Q)$$

and so O and Q are independent.

Q3. We have $|S| = 52$, $|A| = 4$, $|R| = |M| = 26$, $|A \cap R| = |A \cap M| = 2$, $|R \cap M| = 13$ and $|A \cap R \cap M| = 1$. Since the deck is thoroughly shuffled we may assume that each card is equally likely to be chosen and so

$$\begin{aligned}
\mathbb{P}(A) &= \frac{4}{52} = \frac{1}{13}, \\
\mathbb{P}(R) = \mathbb{P}(M) &= \frac{26}{52} = \frac{1}{2}, \\
\mathbb{P}(A \cap R) = \mathbb{P}(A \cap M) &= \frac{2}{52} = \frac{1}{26}, \\
\mathbb{P}(R \cap M) &= \frac{13}{52} = \frac{1}{4}, \\
\mathbb{P}(A \cap R \cap M) &= \frac{1}{52}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbb{P}(A)\mathbb{P}(R) &= \frac{1}{13} \times \frac{1}{2} = \frac{1}{26} = \mathbb{P}(A \cap R) \\
\mathbb{P}(A)\mathbb{P}(M) &= \frac{1}{13} \times \frac{1}{2} = \frac{1}{26} = \mathbb{P}(A \cap M) \\
\mathbb{P}(R)\mathbb{P}(M) &= \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \mathbb{P}(R \cap M) \\
\mathbb{P}(A)\mathbb{P}(R)\mathbb{P}(M) &= \frac{1}{13} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{52} = \mathbb{P}(A \cap R \cap M)
\end{aligned}$$

which are precisely the conditions we need for A, R, M to be mutually independent.

Q4(a) Let A be the event "they have two girls" and B be the event "their first child is a boy".

(i) We want $\mathbb{P}(B|A)$. By definition

$$\mathbb{P}(B|A) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(A)}.$$

We worked out $\mathbb{P}(A \cap B)$ in Q1(d) and $\mathbb{P}(A)$ in Q1(c). So

$$\mathbb{P}(B|A) = \frac{1/4}{11/16} = \frac{4}{11}.$$

(ii) The event "their first child is a girl" is B^c. The questions asks for $\mathbb{P}(B^c|A)$ which by definition is given by

$$\mathbb{P}(B^c|A) = \frac{\mathbb{P}(A \cap B^c)}{\mathbb{P}(A)}.$$

Now

$$\mathbb{P}(A \cap B^c) = \mathbb{P}(\{gg, gbg, gbbg\} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$$

and so

$$\mathbb{P}(B^c|A) = \frac{7/16}{11/16} = \frac{7}{11}.$$

(b)

$$\begin{aligned}
\mathbb{P}(A^c|B) &= \frac{\mathbb{P}(A^c \cap B)}{\mathbb{P}(B)} \quad \text{(by definition)} \\
&= \frac{\mathbb{P}(B) - \mathbb{P}(A \cap B)}{\mathbb{P}(B)} \quad \text{(by axiom 3)} \\
&= 1 - \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\
&= 1 - \mathbb{P}(A|B) \quad \text{(by definition)}
\end{aligned}$$

Suppose A and B are independent. Then $\mathbb{P}(A|B) = \mathbb{P}(A)$ and

$$\mathbb{P}(A^c|B) = 1 - \mathbb{P}(A|B) = 1 - \mathbb{P}(A) = \mathbb{P}(A^c)$$

Hence A^c and B are independent.
The statement that $\mathbb{P}(A^c|B) = 1 - \mathbb{P}(A|B)$ is illustrated in Q4(a) (i) and (ii).
The statement that A^c and B are independent if A and B are independent is illustrated in Q2(b) and (c).

Please let me know if you have any comments or corrections