MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 4

Q1. Let $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Choosing the PIN involves making an ordered choice of 4 elements from X with repetition allowed. Hence the sample space is $S = X^4$ (that is the set of all ordered 4-tuples whose entries are elements of X). There are $10^4 = 10000$ of these so $|S| = 10000$ (of course this is just the same as the number of integers between 0 and 9999).

(a) The number of ways of choosing the PIN to have all even digits is 5^4 (we have five choices for the first digit, five choices for the second digit, and so on). Hence the probability is $5^4/10^4 = (1/2)^4 = 1/16 = 0.0625$.

(b) If we do not allow repetition we have only $10 \times 9 \times 8 \times 7$ choices for the PIN. Hence the probability of this is $10 \times 9 \times 8 \times 7/10^4 = 63/125 = 0.504$.

(c) The PIN is palindromic means it is of the form $xyyx$. So once we have chosen the first 2 digits (which we can do in 10^2 ways) the PIN is completely determined. Hence the probability is $10^2/10^4 = 1/100 = 0.01$.

(d) The number of ways of choosing the PIN to have all digits in $\{0, 1, 2, 3, 4, 5, 6, 7\}$ is 8^4 (there are 8 choices for each of the 4 digits). Hence the probability is $8^4/10^4 = (4/5)^4 = 256/625 = 0.4096$.

(e) If the digits are to be in *strictly* increasing order then there can be no repetitions and each *unordered* selection of 4 distinct digits occurs exactly once. It follows that exactly $\binom{10}{4} = \frac{10\times9\times8\times7}{4\times3\times2\times1} = 10 \times 3 \times 7$ PIN's have this property. The probability is therefore $210/10000 = 0.021$.

Q2.

(a) This is unordered selection without replacement. The sample space has cardinality $\binom{18}{11}$ (this is the number of ways of choosing 11 playes from the squad of 18 players). The number of ways of picking 6 batsmen and 5 bowlers is $\binom{10}{6}\binom{8}{5}$ (the first term being the number of ways of choosing the 6 batsmen from the 10 batsmen in the squad, the second term being the number of ways of choosing the 5 bowlers from the 8 bowlers in the squad). So

$$\mathbb{P}(\text{the team contains 6 batsmen and 5 bowlers}) = \frac{\binom{10}{6}\binom{8}{5}}{\binom{18}{11}}.$$

It is reasonable to leave your answer in this form but you might want to work it out numerically. The key thing to remember here is to cancel things as much as possible:

$$\frac{\binom{10}{6}\binom{8}{5}}{\binom{18}{11}} = \frac{\frac{10!}{6!4!}\frac{8!}{5!3!}}{\frac{18!}{11!7!}}$$
$$= \frac{\frac{10\times9\times8\times7}{4\times3\times2}\frac{8\times7\times6}{3\times2}}{\frac{18\times17\times16\times15\times14\times13\times12}{7\times6\times5\times4\times3\times2}}$$
$$= \frac{10\times3\times7\times8\times7}{18\times17\times8\times13}$$
$$= \frac{5\times7\times7}{3\times17\times13}$$
$$= \frac{245}{663}$$

(b) To find this event we use the fact that $\mathbb{P}$(the team contains fewer than 3 bowlers) is equal to the sum of $\mathbb{P}$(the team contains 10 batsmen and 1 bowler) and $\mathbb{P}$(the team contains 9 batsmen and 2 bowlers). Each of these probabilities can be worked out as in part (a). (I'll leave the details to you this time).

$$\mathbb{P}(\text{the team contains fewer than 3 bowlers}) = \frac{\binom{10}{10}\binom{8}{1}}{\binom{18}{11}} + \frac{\binom{10}{9}\binom{8}{2}}{\binom{18}{11}} = \frac{2}{221}$$

Q3.

(a) Choosing a function means choosing a value in the codomain for each element in the domain (with repetition allowed). We have 3 choices for $f(1)$ and 3 choices for $f(2)$. Hence $|S| = 3 \times 3 = 9$.

For an injective function we don't allow repetition. The number of such functions is therefore $3 \times 2 = 6$ and the probability is $6/9 = 2/3$.

(b) The same argument as for (a) (only this time choosing a sequence of 3 elements from a 2 element set) gives $|S| = 2^3 = 8$.

There are no injections since the domain is larger than the codomain (Lemma 4.1) and so this probability is 0.

(c) The same argument as for (a) (only this time choosing a sequence of m elements from an n element set) gives $|S| = n^m$

For an injective function we don't allow repetition. If $m > n$ then there are no such functions so the probability is 0. If $m \leq n$ then the number of such functions is $n \times (n-1) \times (n-2) \times \cdots \times (n-m+1) = \frac{n!}{(n-m)!}$. So we have

$$\mathbb{P}(h \text{ is injective}) = \begin{cases} 0 & \text{if } m > n \\ \frac{n\times(n-1)\times(n-2)\times\cdots\times(n-m+1)}{n^m} & \text{if } m \leq n \end{cases}$$

Q4. It is simplest to assume that the ten coins are all distinct. Let

$$X = \{C_1, C_2, S_1, S_2, S_3, G_1, G_2, G_3, G_4\}$$

be the set of coins in the bag.

(a) (i) $S = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in X \text{ for } 1 \le i \le 5\}$, or, equivalently, $S = X^5$. We have $|S| = |X|^5 = 9^5$.

(ii) We need to count the number of sequences of five elements of X with exactly two silvers and three golds. We first note that there are $\binom{5}{2} = 10$ ways to choose the positions for the two silver coins. For any such choice there are $3 \times 3 \times 4 \times 4 \times 4$ sequences with the silver coins in these positions and golds in the remaining positions, since we have 3 choices for each silver coin and 4 choices for each gold coin. Thus the total number of sequences with exactly two silvers and three golds is $10 \times 3^2 \times 4^3$. Hence

$$\mathbb{P}(\text{choose two silvers and three golds}) = \frac{10 \times 3^2 \times 4^3}{9^5} = \frac{640}{6561}$$

(b) (i) $T = \{(x_1, x_2, x_3, x_4, x_5) : x_i \in X \text{ for } 1 \le i \le 5 \text{ and } x_i \ne x_j \text{ for } i \ne j\}$. We have $|T| = 9 \times 8 \times 7 \times 6 \times 5$.

(ii) We need to count the number of sequences of five distinct elements of X with exactly two silvers and three golds. As in (a) there are $\binom{5}{2} = 10$ ways to choose the positions for the two silver coins. For any such choice there are $3 \times 2 \times 4 \times 3 \times 2$ sequences with the silver coins in these positions and golds in the remaining positions, since we have 3 choices for the first silver coin, then 2 choices for the second silver, and so on. Thus the total number of sequences with exactly two silvers and three golds is $10 \times 3 \times 2 \times 4 \times 3 \times 2$. Hence

$$\mathbb{P}(\text{choose two silvers and three golds}) = \frac{10 \times 3 \times 2 \times 4 \times 3 \times 2}{9 \times 8 \times 7 \times 6 \times 5} = \frac{2}{21}$$

(c) (i) $U = \{Y \subseteq X : |Y| = 5\}$. We have

$$|U| = \binom{9}{5} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2 \times 1} = 9 \times 7 \times 2$$

(ii) We need to count the number of subsets of X which contain exactly two silvers and three golds. We can construct such a subset by first choosing two silver coins (there are $\binom{3}{2} = 3$ ways to do this), then choosing three gold coins (there are $\binom{4}{3} = 4$ ways to do this). So the number of subsets of X which contain exactly two silvers and three golds is 3×4. Hence

$$\mathbb{P}(\text{choose two silvers and three golds}) = \frac{3 \times 4}{9 \times 7 \times 2} = \frac{2}{21}$$

(d) The probabilities are the same because the event in (b)(ii) does not depend on the order the coins are chosen.

Please let me know if you have any comments or corrections