MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 2

Q1.

(a) This is not a function from A to B since $f(1) = 0 \notin B$.

(b) This is a function from A to B since the given rule defines a unique element of B for each element of A.

(c) This is a function, each real number is mapped to a single real number. Note that this is just one function even though it is defined by three different expressions depending on which part of A we are in.

(d) This is not a function since it doesn't assign a single value to 1 (the second line of the definition suggests that $f(1)$ should be 1, the third line suggests it should be 0). We say in this case that f is not well-defined at 1. Note that there is no such problem at 0; the first and second lines of the definition both define $f(0)$ to be 0.

Q2. Note that a single violation of the definition is enough to show that a function is *not* injective/surjective but a short argument is needed to show that a function is injective/surjective.

(a) This is injective (if $f(x_1) = f(x_2)$ then $3x_1 + 4 = 3x_2 + 4$ and so $x_1 = x_2$).

It is not surjective (no element of $\mathbb{N}$ maps to 1).

It is not bijective and so does not have an inverse.

(b) This is injective (again if $f(x_1) = f(x_2)$ then $3x_1 + 4 = 3x_2 + 4$ and so $x_1 = x_2$).

It is surjective (for any $y \in \mathbb{R}$ we have $\frac{y-4}{3} \in \mathbb{R}$ and $f(\frac{y-4}{3}) = y$).

It is bijective. Since it is bijective it has an inverse function, g, given by $g(y) = \frac{y-4}{3}$.

(c) This is not injective ($f(0) = f(\pi)$).

It is surjective ($\sin(x)$ takes all values in the interval $[-1, 1]$).

It is not bijective and so does not have an inverse.

(d) Drawing a quick sketch of this is probably helpful if you haven't done so yet. Notice that $f(x) \geq 0$ if and only if $x \geq 0$.

The function is injective (if $f(x_1) = f(x_2) \geq 0$ then $x_1^2 = x_2^2$ and $x_1, x_2 \geq 0$ so $x_1 = x_2$, if $f(x_1) = f(x_2) < 0$ then $-x_1^2 = -x_2^2$ and $x_1, x_2 < 0$ and so $x_1 = x_2$).

It is surjective (any non-negative $y \in \mathbb{R}$ is the value of f at $x = +\sqrt{y}$, any negative $y \in \mathbb{R}$ is the value of f at $x = -\sqrt{-y}$.

It is bijective. Since it is bijective it has an inverse function, g, given by

$$g(y) = \begin{cases} +\sqrt{y} & \text{if } y \geq 0 \\ -\sqrt{-y} & \text{if } y < 0 \end{cases}$$

Q3*(a) We first show that f is injective. Suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{N}$. Consider the case when $f(x_1) = f(x_2) \geq 0$. The definition of f implies that x_1 and x_2 must both be even and we have $f(x_1) = x_1/2$ and $f(x_2) = x_2/2$. Thus $x_1/2 = x_2/2$ and so $x_1 = x_2$. The case when $f(x_1) = f(x_2) < 0$ is similar. The definition of f implies that x_1 and x_2 must both be odd in this case, so $f(x_1) = -(x_1+1)/2$ and $f(x_2) = -(x_2+1)/2$. Thus $-(x_1+1)/2 = -(x_2+1)/2$ and we again have $x_1 = x_2$. Thus f is injective.
We next show that f is surjective. Choose $y \in \mathbb{Z}$. If $y \geq 0$ then $2y \in \mathbb{N}$ and $f(2y) = y$. If $y < 0$ then $-2y-1 \in \mathbb{N}$ and $f(-2y-1) = y$. In both cases we can find an element of $\mathbb{N}$ which f maps onto y. Thus f is surjective.
Since f is both injective and surjective, f is bijective.

(b) Let $g : \mathbb{Z} \to \mathbb{N}$ be defined by

$$g(y) = \begin{cases} 2y & \text{if } y \geq 0, \\ -2y-1 & \text{if } y < 0. \end{cases}$$

Choose $x \in \mathbb{N}$. When x is even we have $g \circ f(x) = g(f(x)) = g(x/2) = x$, since $x/2 \geq 0$. When x is odd we have $g \circ f(x) = g(f(x)) = g(-(x+1)/2) = x$, since $-(x+1)/2 < 0$. Thus $g \circ f(x) = x$.
Choose $y \in \mathbb{Z}$. When $y \geq 0$ we have $f \circ g(y) = f(g(y)) = f(2y) = y$, since $2y$ is even. When $y < 0$ we have $f \circ g(y) = f(g(y)) = f(-2y-1) = y$, since $-2y-1$ is odd. Thus $f \circ g(y) = y$.
Hence g is an inverse function to f.

Please let me know if you have any comments or corrections