MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 11

Q1.
As with our examples from lectures, we calculate the the cumulative distribution function first. Clearly T lies between 0 and $\pi/4$ and if $0 \leq t \leq \pi/4$ then $T \leq t$ precisely when p lies within the right-angled triangle with vertices at $(0,0),(1,0),(1,\tan(t))$ (draw a picture to see why). If we call this triangle S_t then

$$\mathbb{P}(T \leq t) = \frac{\text{Area}(S_t)}{\text{Area}(S)} = \frac{\frac{1}{2}\tan(t)}{\frac{1}{2}} = \tan(t).$$

In full the cumulative distribution function is

$$F_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ \tan(t) & \text{if } 0 \leq t \leq \pi/4 \\ 1 & \text{if } \pi/4 < t. \end{cases}$$

To find the probability density function we differentiate (writing $\tan(t) = \sin(t)/\cos(t)$ and using the rule for the derivative of a quotient) obtaining:

$$f_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{\cos^2(t)} & \text{if } 0 \leq t \leq \pi/4 \\ 0 & \text{if } \pi/4 < t. \end{cases}$$

Q2.
(a) We have

$$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } x > b \end{cases}$$

and

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

So

$$\mathbb{E}(X) = \int_a^b \frac{x}{b-a}\,dx = \left[\frac{x^2}{2(b-a)}\right]_a^b = \frac{b+a}{2}$$

Furthermore

$$\mathbb{E}(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \left[\frac{x^3}{3(b-a)}\right]_a^b = \frac{b^2+ab+a^2}{3}$$

so

$$\text{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = \frac{b^2+ab+a^2}{3} - \frac{(b+a)^2}{4} = \frac{b^2-2ab+a^2}{12} = \frac{(b-a)^2}{12}$$

(b) We have

$$F_T(x) = 1 - e^{-\lambda x} \text{ if } x \geq 0 \text{ and } F_T(x) = 0 \text{ if } x < 0.$$

so

$$f_T(t) = \begin{cases} 0 & \text{if } t < 0 \\ \lambda e^{-\lambda t} & \text{if } t > 0. \end{cases}$$

Using integration by parts and the fact that $\mathbb{E}(X) = \lambda^{-1}$ we have

$$\mathbb{E}(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x}\, dx = -\left[x^2 e^{-\lambda x}\right]_0^\infty + 2\int_0^\infty x e^{-\lambda x}\, dx = 0 + 2\lambda^{-1}\mathbb{E}(X) = 2\lambda^{-2}.$$

Thus

$$\operatorname{Var}(X) = \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 2\lambda^{-2} - \lambda^{-2} = \lambda^{-2}.$$

Q3

Let $V = U + 1$. As the hint suggests let's work out the cumulative distribution function of V.

$$\begin{aligned} F_V(x) &= \mathbb{P}(V \leq x) \\ &= \mathbb{P}(U + 1 \leq x) \\ &= \mathbb{P}(U \leq x - 1) \\ &= F_U(x - 1). \end{aligned}$$

We know the cumulative distribution function of U (since if it Uniform$[0,1]$) and so we have

$$F_V(x) = \begin{cases} 0 & \text{if } x - 1 < 0 \\ x - 1 & \text{if } 0 \leq x - 1 \leq 1 \\ 1 & \text{if } x - 1 > 1. \end{cases}$$

or equivalently

$$F_V(x) = \begin{cases} 0 & \text{if } x < 1 \\ x - 1 & \text{if } 1 \leq x \leq 2 \\ 1 & \text{if } x > 2. \end{cases}$$

This is the cumulative distribution function of a Uniform$[1,2]$ random variable and so V (that is $U + 1$) does indeed have a uniform distribution.

Let $W = U^2$. Then W takes values in $[0,1]$ so if it does have a uniform distribution then it must be Uniform$[0,1]$. However

$$\begin{aligned} \mathbb{P}(W \leq 1/2) &= \mathbb{P}(U^2 \leq 1/2) \\ &= \mathbb{P}(-\sqrt{1/2} < U < \sqrt{1/2}) \\ &= F_U(\sqrt{1/2}) - F_U(-\sqrt{1/2}) \\ &= \sqrt{1/2} - 0 \\ &= \sqrt{1/2} \\ &\neq 1/2 \end{aligned}$$

So W does not have the Uniform$[0,1]$ distribution since $F_W(1/2) \neq F_U(1/2)$.

This is enough to show that W is not uniform but if you wanted to work out the full cumulative distribution function you could have done so. It comes out to be:

$$F_W(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sqrt{x} & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } x > 1. \end{cases}$$

This is plainly not the cumulative distribution function of a Uniform$[0, 1]$ random variable.

AQ2.

$$F_T(x) = \begin{cases} 0 & \text{if } x < 0 \\ x^2/2 & \text{if } 0 \leq x < 1 \\ 1 - (2-x)^2/2 & \text{if } 1 \leq x < 2 \\ 1 & \text{if } 2 \leq x \end{cases}$$

$$f_T(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leq x < 1 \\ 2 - x & \text{if } 1 \leq x < 2 \\ 1 & \text{if } 2 \leq x \end{cases}$$

Please let me know if you have any comments or corrections