MTH4107 Introduction to Probability – 2010/11

Solutions to Exercise Sheet 10

Q1.

(a) The cumulative distribution function can be worked out using the fact that for discrete random variables

$$F_A(x) = \sum_{i \le x} \mathbb{P}(A = i)$$

where the sum runs over all $i \le x$ which belong to the range of A.

We get:

$$F_A(x) = \begin{cases} 0 & \text{if } x < -1 \\ 1/10 & \text{if } -1 \le x < 0 \\ 7/10 & \text{if } 0 \le x < 1 \\ 9/10 & \text{if } 1 \le x < 2 \\ 1 & \text{if } 2 \le x \end{cases}$$

Note that it is important to distinguish between $<$ and $\le$ here.

(b) The cumulative distribution function has discontinuities (that is it "jumps") at $r = 1/2, 3, 5$ (draw the graph to see this). This means that the range of B is $\{1/2, 3, 5\}$. So we can work out the probability mass function as:

$$\begin{aligned} \mathbb{P}(B = 1/2) &= \mathbb{P}(B \le 1/2) = F_B(1/2) = 1/5 \\ \mathbb{P}(B = 3) &= \mathbb{P}(B \le 3) - \mathbb{P}(B \le 1/2) = F_B(3) - F_B(1/2) = 3/5 - 1/5 = 2/5 \\ \mathbb{P}(B = 5) &= \mathbb{P}(B \le 5) - \mathbb{P}(B \le 3) = F_B(5) - F_B(3) = 1 - 3/5 = 2/5. \end{aligned}$$

Or equivalently:

n	$1/2$	3	5
$P(B = n)$	$1/5$	$2/5$	$2/5$

We can check the answer by verifying that the values the probability mass function takes add up to 1.

Q2.

(a) To find the probability density function we differentiate the cumulative distribution function (it is differentiable at every real number $r \ne 0, 1$). We obtain

$$f_X(r) = \begin{cases} 0 & \text{if } r < 0 \\ r^{-1/2}/2 & \text{if } 0 < r < 1 \\ 0 & \text{if } 1 < r \end{cases}$$

We could also use $0 \le r \le 1$ as the middle range. As discussed in lectures it is sometimes tidier to have the probability density function defined everywhere. Since we integrate when we use the probability density function it will not make any difference if we change its value at only a finite number of points.

(b) This involves integrating. The limits of integration are 0 and 1 because the probability density function is 0 outside this range.

$$
\begin{aligned}
\mathbb{E}(X) &= \int_0^1 r f_X(r) dr = \int_0^1 r^{1/2}/2\, dr = \left[r^{3/2}/3\right]_{r=0}^{r=1} = 1/3 \\
\mathbb{E}(X^2) &= \int_0^1 r^2 f_X(r) dr = \int_0^1 r^{3/2}/2\, dr = \left[r^{5/2}/5\right]_{r=0}^{r=1} = 1/5 \\
\mathrm{Var}(X) &= \mathbb{E}(X^2) - \mathbb{E}(X)^2 = 1/5 - (1/3)^2 = 4/45
\end{aligned}
$$

(c) The lower quartile is the number c which satisfies $F_X(c) = \sqrt{c} = 1/4$. So the lower quartile is $1/16$.

AQ1.

(a) The cumulative distribution function is

$$
F_Y(r) = \begin{cases} 0 & \text{if } r < 1 \\ \frac{1}{2}(r-1)^2 & \text{if } 1 \le r < 2 \\ 1 - \frac{1}{2}(3-r)^2 & \text{if } 2 \le r < 3 \\ 1 & \text{if } 3 \le r \end{cases}
$$

Note that in this case (as the function is continuous) it doesn't matter if, for example, we have < 1 or ≤ 1 in the first line of the definition of $F_Y(r)$.

(b) You could check (some or all of) the following: that the function is continuous, that it is non-decreasing, that it takes values between 0 and 1 only, that $F_Y(t) \to 1$ as $t \to \infty$ and that $F_Y(t) \to 0$ as $t \to -\infty$.

(c)

$$
\mathbb{P}(3/2 < Y < 5/2) = \frac{3}{4}.
$$

(d)

$$
\mathbb{P}(3/2 < Y < 5/2) = \frac{3}{4}.
$$