Are We Encouraging Our Students to Think Mathematically?

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Introduction

In her 1990 paper, *Pedagogy and the Disciplines*, Ursula Wagener, of the University of Pennsylvania, describes a mathematics class:

> *A graduate student teacher in a freshman calculus class stands at the lectern and talks with enthusiasm about how to solve a problem: "Step one is to translate the problem into mathematical terms; step two is ..." Then she gives examples. Across the room, undergraduates memorize a set of steps. Plugging and chugging—teaching students how to put numbers into an equation and solve it—elbows out theory and understanding.*

The teacher in this classroom knows the subject matter. Her delivery and pacing are impeccable. Yet she teaches mathematics as a bag of tricks, rather than as an understanding of fundamental principles.

Is this an accurate description of the mathematics classes that we are teaching?

In order to help students learn to think mathematically, we first need to understand how they are thinking. Do students demand "plug and chug" from us? It is not so much that they demand it as they expect it. If we want to change their expectations, we must first understand their view of mathematics. Many students come to college with attitudes and expectations that may startle us.

Students' Expectations of First-Year Math Courses

One window into students' thinking processes is their comments on end-of-semester questionnaires. There are frequent comments about the instructor's clarity (or lack of it). Accessibility, sense of humor and accent are also important to students. They are also deeply concerned, and sometimes incensed, by issues of equity. Within a single course with common exams, some instructors are better than others, some give more handouts than others, some give review problems closer to the exam questions than others.

Another common complaint is that the instructor or the text didn't give the students enough help with the homework problems. For example:[1] "We spent hours doing some problems which [the instructor] didn't tell us how to do." Several would agree with the student who said the thing he liked least was that "...the examples provided did not help with homework." Another suggested, "I wish we could have gone over one example like the homework problems ..."

[1]Source: Course evaluations.

before we did the homework." Since many students didn't read math texts much in high school, but simply used them as a source of problems and examples, the kind of help they are asking for is a worked example, given in class or the text, which *closely* parallels the homework problem. To confirm this, the students in two courses at Harvard were surveyed and asked whether or not they agreed with the statement:

> *If you can't do a homework problem you should be able to find a worked example in the text that will show you how.*

Students in calculus gave it 4.1 out of 5 (where 5 indicates strong agreement); those in precalculus gave it a 4.7 out of 5. Further evidence of students' expectations came from the student who suggested that review problems should have the relevant section of the text listed after them in parentheses. When surveyed, his classmates agreed (4.2 out of 5 for calculus, 4.8 out of 5 for precalculus). More explicit still was the student who remarked that the best thing about the teaching in his section was that things were explained "in a cookbook fashion."

As instructors, we can learn a lot about how students think by listening to what they find difficult. The following two examples reminded me that although I feel it is important to know the meaning of one's computations, I do not always succeed in getting my point across.

- **Example: Wanting a 'Step-by-Step Approach.'** A student who had done badly on a Calculus II hour exam came to me complaining that he needed more step-by-step instructions on how to do the problems. Trying to narrow down the request, I asked what topics he already knew. One of them he was sure he knew, he said, was Euler's method. He had recently earned full credit on the following exam questions:

  > Consider the differential equation $dy/dx = x^2 + y$. **(a)** Use Euler's method with two steps to approximate the value of y when $x = 2$ on the solution curve that passes through (1,3). Explain clearly what you are doing on a sketch. Your sketch should show the coordinates of all the points you have found. **(b)** Are your approximate values of y an under- or over- estimate? Explain how you know.

  To check, I asked the student if he could draw a picture of the calculation he did for Euler's method. After a bit of thought, he said yes, he was sure he could draw such a picture. To check still further, I asked him what Euler's method was calculating. There was a *long* silence. Finally he said, rather hesitantly, that he thought it was the arc-length—the arc length along the polygonal curve.

Thus I would suggest that a more step-by-step method is not what was needed—but rather much greater attention on the part of both the instructor and the student to the meaning of the computations being performed. We should realize that exam problems which start, "Use such-and-such a method to do such-and-such," may not be testing all that we want to test. What if this exam

question had been worded, "If $dy/dx = x^2 + y$ and $y(1) = 3$, estimate $y(2)$"? However, rewording this problem in this way would have caused some complaints, as students clearly agree (4.1 out of 5 in calculus and 4.6 out of 5 in precalculus) with the statement that:

> *A well-written problem makes clear what method to use to solve it.*

- **Example: 'Vaguely Worded' Problems.** Another student doing badly in Calculus II came to me after an exam to find out what to do. The problem was, as he described it, that although he understood the basic ideas, he couldn't apply them because of the "vague" way in which the problems were worded. The example he chose to illustrate this was the following question:

  > Alice starts at the origin and walks along the graph of $y = x^2/2$ in the positive x-direction at a speed of 10 units/second. **(a)** Write down the integral which shows how far Alice has traveled when she reaches the point where $x = a$. **(b)** You want to find the x-coordinate of the point Alice reaches after traveling for 2 seconds. Find upper and lower estimates, differing by less than 0.2, for this coordinate. Explain your reasoning carefully.

  The student had been unable to do this question because he hadn't realized that it was about arc length. He felt quite strongly that the wording of the question should have mentioned arc length specifically.

Again, the problem here seems to be how to teach students to do problems which do not explicitly ask for a certain computation, as well as how to get them to believe that such problems are reasonable.

How Do Students' Expectations Develop?

Most students come to college expecting new experiences. At a residential institution, we need to constantly remind ourselves that many students have never lived away from home before; many have never met students from different backgrounds, religions, or ethnic groups. It is easy to forget that many students have no idea what anthropology is and have never had the chance to take psychology or economics or visual arts. Most freshmen are eager for new experiences, but they are also insecure. Are they the admissions office's one mistake? Is it really worth their family making the financial sacrifices that are required to send them to college? One of the best ways for freshmen to reassure themselves that they are "doing OK" is to get decent grades—which often means *very* good grades, since they are often used to all A's. This makes them want to take some courses which are familiar and predictable so that they can be sure they will do well. This is particularly true in math and science where the grading and workload are inclined to be tougher than in other fields.

Undergraduates are busy people. It is easy for us to underestimate the pressure to join extracurricular activities and our students need to hold a job (or even two). Consequently, undergraduates prize courses and instructors which do not take too much of their time outside of class. A course in which they are expected to read about some topics on their own may have students struggling to get out of it, not because they can't do the work but because it takes too much time and they don't want to have to "teach the course to themselves." The feeling that everything "ought" to be covered in class is reflected in student reactions to mathematics. As an example, one student said that the best thing about the teaching in his section was the fact that

> *It was possible for me to learn the material without studying on my own too much if I paid attention in class.*

Thus most undergraduates, like most faculty, have more to do than they can reasonably fit into their schedules and tend to look for short-cuts. As an analogy for the way in which some students go about learning mathematics, imagine what it takes to become proficient at an office manager's job. Most undergraduates regard learning mathematics in much the same way a mathematician might regard learning such a job: as something you have to be shown how to do.

This view of mathematics is partly reasonable—we don't expect students to come up with the idea of a derivative by themselves, or to figure out the Fundamental Theorem of Calculus on their own. However, carried to extremes it can lead to absurd results. Consider, for example, the student (who had already taken BC Calculus) who was complaining rather loudly that he shouldn't be asked to suggest formulas for functions whose graphs he had been given. He said that he "did graphs" in the other order: If we gave him the formula, he'd draw the graph, but he wasn't doing this. When it was suggested that he might need to find formulas to fit lab data in his intended major (chemistry), he announced that he'd done experiments before and that one didn't *ever* need to do such a thing. It is important to realize that the vehemence that leads students to dig in their toes as completely as this one is born, at least partly, out of terror, not out of a real desire to be closed-minded.

A certain amount of predictability is entirely appropriate and necessary. Too much, however, means that it is easier for students to memorize how to do a long list of "types" of problems than to learn the basic ideas of mathematics. Many of our students are sufficiently diligent and sufficiently scared of not performing well that they will willingly master a very long set of problem-solving procedures, even by memorization. For example, while trying to figure out how to prepare for a calculus final, one student asked me if he should do the review problems over and over again until he could do them without looking at the solutions. In this way they differ greatly from the "creatively lazy" professional mathematician who would much rather figure out how it all works than memorize anything.

Most students coming to college know, at least in theory, that learning mathematics involves developing understanding. Many of them, often on their own, have developed an understanding of some topics. However, few of them have actually taken courses where an understanding was really *required*—in virtually every case, it was possible to do well just by learning to do all the types of problems shown in class. Consequently, asking students to do problems which

have not been modeled for them in class or in the reading is inclined to strike them as unfair.

The Instructor's Role

There is ample evidence that the students' views of mathematics are often startlingly different than ours when they start college. What can be done about it? Is encouraging students to think mathematically part of our job, or should we just work with those students who are naturally inclined this way?

I suggest that we may be letting our students down if we do not try to broaden and deepen their thinking. It is, of course, much easier for us to teach a course in which we focus on template problems and algorithms. It's even possible to put the burden on the students by saying that they won't accept anything else. But is that really true? Perhaps, as George Rosenstein, Franklin and Marshall, suggests,[2] we have played a role in allowing these expectations to become established:

> *I'm convinced there has been a conspiracy between math teachers and math students. The terms are that the teachers can do whatever they want in class, but will ask for only the well-practiced or routine on the exams. In return, the students will be cooperative and diligent at learning the manipulative or template material that is stressed on homework assignments and quizzes.*

Our biggest challenge as teachers is to understand our students' thinking patterns well enough that we can affect them. Learning to think more independently is a difficult, often frightening, process for students. Thus, besides gaining an understanding of our students' thinking, we need to understand their feelings well enough to gain their trust.

How Should Our Students' Views of Mathematics Affect Our Teaching?

Before considering how to react to our students' view of mathematics, we need to consider what is meant by understanding. Here, too, there is a difference between professional mathematicians and first year undergraduates. In most students' minds, understanding is being able to visualize (or otherwise internally represent) concepts and the relationships between them. The notion which mathematicians might call, or include in, understanding—knowing the theoretical and logical connections between concepts—is not what most students mean by understanding. For most students, the visual form of understanding precedes the more theoretical version. Thus, to take students toward a rigorous thinking, we must first establish a solid intuitive, visual understanding.

Each department and each faculty member needs to decide exactly how to approach students' views of mathematics. Not to challenge these views at

[2] From Project CALC Newsletter, October 1991.

all is not doing our students justice, as well as not in the best interests of the profession. However, challenging them too much alienates students from the subject and teaches them little. The most useful guide to how much to challenge them is a robust understanding of students. To acquire this, each instructor is well advised to use every course they teach as an opportunity to learn about their students. Some techniques that may be useful:

1. Include problems on tests that ask students to sketch or explain. Grade them yourself, or look them over before handing them back.
2. Ask students to read some mathematics and summarize what they have read in a paragraph—and find time to read, or skim, the paragraphs.
3. Arrange an e-mail discussion group for a course. Look over the postings every few days.
4. A particularly good way to gain insight into what students are thinking is the "one-minute paper," advocated by Richard Light at Harvard. At the end of class ask students to take a piece of paper and write on it:
   - The most important thing learned that day.
   - The most confusing thing that day.
   - One question they have that remains unanswered.

   The pieces of paper are handed in as the students leave the room; the instructor reads through them before planning the next class. The answers to these questions can often be illuminating—and occasionally devastating—to the instructor. It becomes easier to see what the students are thinking, and addressing their misconceptions becomes more urgent. Acknowledging and acting on the responses can markedly improve communication with the class.

Conclusion

Successful teaching involves knowing what your students are thinking. However, it is often hard for instructors to "hear" students—especially as the students' thoughts are often not similar to their own. Consequently, any efforts to listen to students' thinking about mathematics are likely to improve one's teaching.

Big Business, Race, and Gender in Mathematics Reform

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The mathematics reform movement may have positive attributes, but that is not what this appendix is about. This essay is divided into three sections, each taking a critical view of what has come to be called "mathematics reform." Rather than attempting an abstract definition of this term, I cite the principal documents and leaders of the reform movement on particular issues. The fault line separating the mathematics reform movement from its critics is nowhere more volatile and portentous than in California. The third and final section of this appendix is devoted to a short history of the conflict over mathematics reform in that state, with a focus on the controversial California mathematics standards. This set of standards has received widespread praise from prominent mathematicians and strong opposition from the mathematics reform community. As explained in the last section, this conflict helps to define, in *practical* terms, the mathematics reform movement.

The second section challenges assumptions about ethnicity and gender in the reform movement. Multiculturalism and mathematics for "all students" are recurring themes among reformers. Prominent reformers claim that learning styles are correlated with ethnicity and gender. But reform curricula, while purporting to reach out to students with different "learning styles", actually limit opportunities. Fundamental topics, including algebra and arithmetic are abridged or missing in reform curricula without apology.

Big Business and the mathematics reform movement have at least one thing in common. They both militate for more technology in the classroom. Calculators and computers are regular features in reform math curricula, and technology corporations routinely sponsor conferences for mathematics educators. The confluence of interests and the resulting momentum in favor of more technology is the subject of the first section.

Technology, Reform, and the Corporate Influence

The 1989 report "Everybody Counts" warned:

> In spite of the intimate intellectual link between mathematics and computing, school mathematics has responded hardly at all to curricular changes implied by the computer revolution. Curricula, texts, tests, and teaching habits—but not the students—are all products of the pre-computer age. Little could be worse for mathematics education than an environment in which schools hold students back from learning what they find natural. [17]

The imperative to integrate technology into the classroom goes far beyond mathematics courses. President Clinton calls for "a bridge to the twenty first century ... where computers are as much a part of the classroom as blackboards."