

Fokker-Planck theory of superstatistics

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Abstract Complex systems are frequently governed by a hierarchy of dynamics on different time scales. Superstatistics, which is a “statistics of statistics” with several largely separated time scales, aims to describe their nonequilibrium stationary states. Here, a prototype system, i.e., a Brownian particle moving through a fluctuating medium with slowly varying temperature, is considered, and a Fokker-Planck theory is developed to establish a dynamical foundation of superstatistics, which includes time evolution. In this theory, both the velocity of the particle and the inverse temperature are dynamically treated with the help of a Born-Oppenheimer-like adiabatic scheme introducing a dynamical hierarchy. It is shown that the Fokker-Planck equation admits then as a general stationary solution a superstatistical one. Three major universality classes often observed in nature, i.e., gamma, inverse gamma, and log-normal superstatistics, are discussed as special solutions.

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Nonequilibrium complex systems are often governed by dynamics with a hierarchical structure composed of different dynamics on different time scales. Decomposition of the total dynamics into various subdynamics may be relatively simple if the time scales characterizing the subdynamics are largely separated. Therefore, considering two random variables, say X and Y , suppose that the dynamics of X is much faster than that of Y . Then, the joint probability distribution, $P(x,y)$, with x and y being the realizations of X and Y , respectively, is conveniently described as follows: $P(x,y) = P(x|y)P(y)$, where $P(x|y)$ is the conditional probability distribution of X , given a value of Y to be y , and $P(y)$ is the marginal distribution of Y . A physical example is a nonequilibrium thermostistical system. It is divided into spatial cells, each of which is in local equilibrium and contains a sufficiently large number of particles. The state of each cell is characterized by a local inverse temperature, $\beta \equiv 1/(k_B T)$ which depends on space and time. The Boltzmann constant is denoted by k_B . If the system is superstatistical, then the local energy of the cell, E , with its i th value of realization, ε_i , is a fast variable, whereas the local inverse temperature, B , with its realization, β , is a slow one. In other words, relaxation of the system to a local equilibrium state at a given value of the inverse temperature, β , characterized by the conditional distribution, $P(\varepsilon_i | \beta) = Z^{-1}(\beta) \exp(-\beta \varepsilon_i)$ with $Z(\beta) = \sum_i \exp(-\beta \varepsilon_i)$, is fast, but β slowly varies according to some distribution, $f(\beta)$. Therefore, B describes a randomness quenched in the relaxation of the system. The energy distribution to be observed is then given by the average of $P(\varepsilon_i | \beta)$ with respect to $f(\beta)$:

$$P(\varepsilon_i) = \int d\beta P(\varepsilon_i | \beta) f(\beta), \quad (1)$$

which is the marginal distribution of E and may describe a nonequilibrium stationary state of the system. It may be radically different from the canonical ensemble, in general, unless $f(\beta)$ is very sharply peaked at a certain value. We remark that both $P(\varepsilon_i)$ and $f(\beta)$ are seen to be the marginal distributions calculated from the joint distribution, $P(\varepsilon_i, \beta) \equiv P(\varepsilon_i | \beta) f(\beta)$.

The framework discussed above is referred to as superstatistics [1]. It has actually been anticipated in Refs. [2-4], but the work in Ref. [1] is the first in which the large separation of the time scales is put as a physical basis of the scheme and general $f(\beta)$ are considered. Recently, the concept of hyperensemble (or, superensemble) has been introduced in Refs. [5-7], which is a related attempt to generalize the Gibbsian equilibrium ensemble to nonequilibrium. We also mention that the connection of superstatistics to nonequilibrium thermodynamics has been established in Ref. [8].

The idea of superstatistics has found a considerable variety of applications, many of which are beyond the scope of traditional thermostatics. Among others, we wish to point out the following: tracer particles in turbulent flows [9], defect turbulence [10], velocity difference in turbulence [11], violent relaxation of collisionless stellar systems [12], complex scale-free networks emerging from fluctuating random graphs [13], ecosystems driven by hydroclimatic fluctuations [14], random-matrix approach to traffic flows [15], multiplicative-noise processes [16], progression of cancer cells [17],

energy spectra of cosmic rays [18], thermal quantum entanglement [19], fluctuating Bose-Einstein condensates of photons [20], and quantum turbulence [21].

A question of central importance about superstatistics is what the relevant procedure is for determining $f(\beta)$. Without clarifying this, Eq. (1) is formal: $f(\beta)$ remains just the inverse Laplace transformation of the observed $P(\varepsilon_i)$. Physically, it is desirable to be able to determine $f(\beta)$ from underlying dynamics to explain $P(\varepsilon_i)$. Actually, there are only few studies on this issue [7,8,22,23]. In Refs. [7,8], the “conditional maximum entropy method” is formulated as a possible principle. Also, it is shown in Ref. [23] how a universality class termed log-normal superstatistics [see Eq. (33) below] is derived from the fluctuation theorems [24-26].

The purpose of this paper is to develop a novel *dynamical* approach to superstatistics. More specifically, we will show that superstatistical distributions can be obtained as stationary solutions of a suitable Fokker-Planck equation that incorporates both the fast and slow variables. To accomplish this, we consider 1-dimensional Brownian motion through a background medium with slowly varying inverse temperature, which is a prototype system in superstatistics [1,3]. In this theory, both the velocity, v , of the particle as a fast variable and the inverse temperature, β , as a slow variable are random variables at the level of a stochastic differential equation. *Not only the velocity but also the inverse temperature is dynamically treated.* Then, we analyze with the help of a Born-Oppenheimer-like adiabatic scheme the Fokker-Planck equation associated with the stochastic differential equation. Using a new method for obtaining a stationary solution of the multivariate Fokker-Planck equation, we explicitly show how $f(\beta)$ is

determined in terms of the drift term in the dynamics of β . We also describe three major universality classes [27], i.e., gamma, inverse gamma, and log-normal superstatistics, as special solutions.

After this introduction, we start our discussion with the consideration of a two dimensional vector-valued random variable

$$\mathbf{X} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \equiv \begin{pmatrix} V \\ B \end{pmatrix}, \quad (2)$$

where V is the velocity of a Brownian particle with unit mass moving through a fluctuating medium with slowly varying inverse temperature, B . The dynamics of \mathbf{X} is governed by a Langevin equation in the form of the following stochastic differential equation [28]:

$$d\mathbf{X} = \mathbf{K}(\mathbf{X}) dt + G d\mathbf{W}. \quad (3)$$

Here, $\mathbf{K}(\mathbf{X})$ is a two-dimensional drift vector and G is a constant 2×2 matrix. $d\mathbf{W}$ is the Wiener noise vector satisfying the Ito rule:

$$dW_i dW_j = \delta_{ij} dt. \quad (4)$$

Thus, not only the velocity but also the inverse temperature is treated as a dynamical variable here.

V is a fast variable, whereas B is a slow one. The adiabatic scheme requires that the

dynamics of B should not depend on V . Therefore, in contrast to K_1 , K_2 must actually be independent of V :

$$K_1 = K_1(V, B), \quad K_2 = K_2(B). \quad (5)$$

This is analogous to the Born-Oppenheimer approximation in quantum mechanics [29] and introduces a *dynamical hierarchy* into Eq. (3).

The probability, $P(\mathbf{x}, t)d^2\mathbf{x}$, of finding the velocity and inverse temperature in the interval between $\mathbf{x} = (v, \beta)$ and $\mathbf{x} + d\mathbf{x} = (v + dv, \beta + d\beta)$ at time t satisfies the Fokker-Planck equation [25]:

$$\frac{\partial P(\mathbf{x}, t)}{\partial t} = - \sum_{i=1}^2 \frac{\partial J_i(\mathbf{x}, t)}{\partial x_i}, \quad (6)$$

where $J_i(\mathbf{x}, t)$ is the probability current density given by

$$J_i(\mathbf{x}, t) = K_i(\mathbf{x})P(\mathbf{x}, t) - \frac{1}{2} \sum_{j=1}^2 D_{ij} \frac{\partial P(\mathbf{x}, t)}{\partial x_j}, \quad (7)$$

and

$$D = GG^T \quad (8)$$

is the *symmetric* diffusion matrix.

Next, let us consider the stationary solution of Eq. (6). The stationarity condition yields

$$\sum_{i=1}^2 \frac{\partial J_i(\mathbf{x})}{\partial x_i} = 0, \quad (9)$$

Frequently discussed are absorbing, reflecting, and periodic boundary conditions [28] on the probability current and/or the probability distribution to solve Eq. (9). But, such an analysis as a boundary-value problem turns out not to be convenient for our purpose. Instead, we propose here the following different approach. In order to satisfy Eq. (9), we set

$$J_i(\mathbf{x}) = \sum_{j=1}^2 \frac{\partial h_{ij}(\mathbf{x})}{\partial x_j}, \quad (10)$$

where $h_{ij}(\mathbf{x})$ is an antisymmetric tensor. In two dimensions, $h_{ij}(\mathbf{x})$ is proportional to the Levi-Civita symbol ($\varepsilon_{ij} = -\varepsilon_{ji}$, $\varepsilon_{12} = 1$)

$$h_{ij}(\mathbf{x}) = \hat{h}(\mathbf{x}) \varepsilon_{ij}. \quad (11)$$

With this choice, Eq. (9) is automatically fulfilled (A generalization of this approach to a higher-dimensional case is straightforward). Without losing generality, we can write

$$\hat{h}(\mathbf{x}) = h(\mathbf{x}) P(\mathbf{x}). \quad (12)$$

Then, Eq. (10) together with Eqs. (7), (11), and (12) leads to the following coupled equations:

$$\left(K_1 - \dot{h} \right) P - \frac{D_{11}}{2} P' - \left(\frac{D_{12}}{2} + h \right) \dot{P} = 0, \quad (13)$$

$$(K_2 + h')P - \frac{D_{22}}{2} \dot{P} - \left(\frac{D_{21}}{2} - h \right) P' = 0, \quad (14)$$

where the primes and dots denote differentiations with respect to v and β , respectively.

Now, our interest is to see if a superstatistical solution

$$P(\mathbf{x}) = \tilde{f}(\beta) \exp(-\beta \varepsilon), \quad \varepsilon = \frac{1}{2} v^2, \quad (15)$$

with

$$\tilde{f}(\beta) = \frac{f(\beta)}{Z(\beta)}, \quad Z(\beta) = \sqrt{\frac{2\pi}{\beta}} \quad (16)$$

can be obtained as a stationary solution of Eq. (6), where $P(v|\beta) \equiv Z^{-1}(\beta) \times \exp(-\beta v^2/2)$ is the one-dimensional Maxwellian distribution for unit particle number density. Superstatistics of the particle velocity is then characterized by the marginal distribution

$$P(v) = \int_0^\infty d\beta P(\mathbf{x}). \quad (17)$$

Substituting Eq. (15) into Eqs. (13) and (14), we obtain

$$\left[h(v, \beta) + \frac{D_{12}}{2} \right] \dot{\tilde{f}}(\beta) = \left[K_1(v, \beta) - \dot{h}(v, \beta) + \frac{1}{2} h(v, \beta) v^2 + \frac{D_{11}}{2} \beta v + \frac{D_{12}}{4} v^2 \right] \times \tilde{f}(\beta), \quad (18)$$

$$\frac{D_{22}}{2} \dot{\tilde{f}}(\beta) = \left[K_2(\beta) + h'(v, \beta) - h(v, \beta) \beta v + \frac{D_{21}}{2} \beta v + \frac{D_{22}}{4} v^2 \right] \tilde{f}(\beta). \quad (19)$$

These are two coupled equations for $\tilde{f}(\beta)$ and $h(v, \beta)$. Consistency between Eq. (18) and (19) requires that there exists a function $A(\beta)$ depending only on the inverse temperature such that

$$K_1(v, \beta) - \dot{h}(v, \beta) + \frac{1}{2} h(v, \beta) v^2 + \frac{D_{11}}{2} \beta v + \frac{D_{12}}{4} v^2 = \left[h(v, \beta) + \frac{D_{12}}{2} \right] A(\beta), \quad (20)$$

$$K_2(\beta) + h'(v, \beta) - h(v, \beta) \beta v + \frac{D_{21}}{2} \beta v + \frac{D_{22}}{4} v^2 = \frac{D_{22}}{2} A(\beta), \quad (21)$$

so that the equation for $\tilde{f}(\beta)$ reads

$$\dot{\tilde{f}}(\beta) = A(\beta) \tilde{f}(\beta). \quad (22)$$

Using Eq. (16), this equation can be rewritten as

$$\dot{f}(\beta) = \left[A(\beta) - \frac{1}{2\beta} \right] f(\beta). \quad (23)$$

Given a particular form of $A(\beta)$, use of the solution of this equation in Eq. (15) establishes superstatistics based on the marginal distribution in Eq. (17).

Differentiating Eq. (21) with respect to v , we have

$$[h'(v, \beta) - h(v, \beta) \beta v]' + \frac{D_{21}}{2} \beta + \frac{D_{22}}{2} v = 0. \quad (24)$$

This equation admits as a solution the following linear function of v :

$$h(v, \beta) = \frac{D_{21}}{2} + \frac{D_{22}}{4\beta} v. \quad (25)$$

Therefore, from Eqs. (20) and (21), the components of the drift vector $\mathbf{K}(\mathbf{x})$ are found to be

$$K_1(v, \beta) = D_{12} A(\beta) + \left[\frac{D_{22}}{4\beta} A(\beta) - \frac{D_{11}}{2} \beta - \frac{D_{22}}{4\beta^2} \right] v - \frac{D_{12}}{2} v^2 - \frac{D_{22}}{8\beta} v^3, \quad (26)$$

$$K_2(\beta) = \frac{D_{22}}{2} \left[A(\beta) - \frac{1}{2\beta} \right], \quad (27)$$

where the symmetry $D_{12} = D_{21}$ has been used.

Interestingly, $K_1(v, \beta)$ is cubic in v . This is in contrast to the original discussion in Ref. [1], in which superstatistics of the form in Eq. (15) is derived from a Langevin equation with drift term linear in v . The main difference of the present theory from that in Ref. [1] is that β is a dynamical quantity here. Accordingly, the components of the probability current density of the velocity and the inverse temperature are not independent but influence each other in a dynamical way via the coupled equations (18) and (19). Thus, the cubic nature of $K_1(v, \beta)$ is a dynamical consequence.

Note that the strength of the drift term for the velocity in the Langevin equation in Eq. (3) also depends on $A(\beta)$ and hence on the inverse temperature distribution. Eq. (27) has physical meaning as well. $A(\beta)$ is essentially the drift term in the Langevin equation for the inverse temperature. This drift is a dynamical quantity that can, in

principle, be directly determined in experiments if temporal changes of the (inverse) temperature are measured.

Finally, let us look at some examples. Here, we discuss three major universality classes as introduced in Ref. [27]: gamma, inverse gamma, and log-normal superstatistics. They are simply defined to be superstatistics where the inverse temperature is gamma, inverse gamma, or log-normally distributed, respectively.

(i) Gamma superstatistics: In this case, $A(\beta)$ takes the form

$$A(\beta) = -\frac{1}{\beta_0} + \frac{p-1/2}{\beta}, \quad (28)$$

where β_0 and p are positive constants. Then, from Eq. (23), $f(\beta)$ is found to be the gamma distribution

$$f(\beta) = \frac{1}{\Gamma(p)\beta_0} \left(\frac{\beta}{\beta_0}\right)^{p-1} e^{-\beta/\beta_0}, \quad (29)$$

where $\Gamma(p)$ is Euler's gamma function. Physical examples where this superstatistics is relevant are e.g., defect turbulence [10] and quantum turbulence [21].

(ii) Inverse gamma superstatistics: $A(\beta)$ is taken to be

$$A(\beta) = -\frac{\beta_0}{\beta^2} - \frac{p+1/2}{\beta}, \quad (30)$$

so that it yields the inverse gamma distribution

$$f(\beta) = \frac{1}{\Gamma(p)\beta_0} \left(\frac{\beta_0}{\beta}\right)^{p+1} e^{-\beta_0/\beta}. \quad (31)$$

This kind of superstatistics describes, for example, cancer survival statistics [17].

(iii) Log-normal superstatistics: Here, $A(\beta)$ is given by

$$A(\beta) = -\frac{\ln(\beta/\beta_0)}{\sigma^2 \beta} - \frac{1}{2\beta}, \quad (32)$$

where σ is a constant. This yields the log-normal distribution

$$f(\beta) = \frac{1}{\beta\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{[\ln(\beta/\beta_0)]^2}{2\sigma^2}\right\}. \quad (33)$$

Log-normal superstatistics is related to classical fully developed turbulence [11].

In conclusion, we have formulated the Fokker-Planck theory of superstatistics by considering the prototype superstatistical system, i.e., a Brownian particle moving through a medium with slowly varying temperature. We have treated both the velocity of the particle and the inverse temperature dynamically. We have realized superstatistics as a stationary solution of the Fokker-Planck equation. We have shown how the adiabatic scheme (i.e., the Born-Oppenheimer-like feature) plays a crucial role in the description of the dynamical hierarchy. While we have mainly concentrated on the existence of stationary solutions in this paper, the advantage of our Fokker-Planck approach is that it also allows for the treatment of more general dynamical problems, for

example the problem how fast the superstatistical stationary state is approached starting from an arbitrary initial condition.

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