

Stretched exponentials from superstatistics

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Abstract

Distributions exhibiting fat tails occur frequently in many different areas of science. A dynamical reason for fat tails can be a so-called superstatistics, where one has a superposition of local Gaussians whose variance fluctuates on a rather large spatio-temporal scale. After briefly reviewing this concept, we explore in more detail a class of superstatistics that hasn't been subject of many investigations so far, namely superstatistics for which a suitable power β^η of the local inverse temperature β is χ^2 -distributed. We show that $\eta > 0$ leads to power law distributions, while $\eta < 0$ leads to stretched exponentials. The special case $\eta = 1$ corresponds to Tsallis statistics and the special case $\eta = -1$ to exponential statistics of the square root of energy. Possible applications for granular media and hydrodynamic turbulence are discussed.

1 Introduction

Nonextensive statistical mechanics [1, 2, 3, 4] was originally developed as an equilibrium formalism, but most physical applications of this formalism actually occur for typical nonequilibrium situations. Sometimes these nonequilibrium situations are described by a fluctuating parameter β , which may, for example, be the inverse temperature. Alternatively, β may be an effective friction constant, a changing mass parameter, a changing amplitude of Gaussian white noise, a fluctuating local energy dissipation or simply a local inverse variance parameter extracted from a time series. The fluctuations of β induce a superposition of different statistics on different time scales, in short a superstatistics [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23]. The stationary probability distributions of superstatistical systems typically exhibit much broader tails than a Gaussian distribution. These tails can decay e.g. with a power law, or as a stretched exponential, or in an even more complicated way [8]. Which type of tails are produced depends on the probability distribution $f(\beta)$ of the parameter β . Recent applications of the superstatistics concept include a variety of physical systems. Examples are Lagrangian [24, 25, 26, 27] and Eulerian turbulence [28, 29, 30], defect turbulence [31], atmospheric turbulence [32, 33], cosmic ray statistics [34], solar flares [35], solar wind statistics [36], networks [37, 38], random matrix theory [39], and mathematical finance [40, 41, 42].

If β is distributed according to a particular probability distribution, the χ^2 -distribution, then the corresponding marginal stationary distributions of the superstatistical system obtained by integrating over all β are given by the generalized canonical distributions of nonextensive statistical mechanics [1, 2, 3, 4]. For other distributions of the intensive parameter β , one ends up with more complicated statistics.

In this paper, after briefly reviewing the superstatistics concept, we explore a rather general case which may be of relevance to many practical applications. We consider the case of a superstatistics where β^η , i.e. β to some power η , is χ^2 -distributed, where η is some arbitrary parameter. The case $\eta = 1$ is fully understood: It leads to Tsallis statistics and asymptotic power-law decay of the marginal distributions obtained by integrating over all β . However, the other values of η are interesting as well, and will be explored in more detail here. For general $\eta > 0$ we obtain asymptotic power law decay, though the resulting statistics is slightly different from Tsallis statistics (only $\eta = 1$ leads exactly to Tsallis statistics). For $\eta < 0$ one obtains

tails that asymptotically decay as stretched exponentials. The special case $\eta = -1$ corresponds to exponential tails of the square root of energy. We will provide some arguments (based on the ordinary Central Limit Theorem) why nonequilibrium systems with many degrees of freedom often lead to one of the superstatistics described above.

2 Various types of superstatistics

It is well known that for the canonical ensemble the probability to observe a state with energy E is given by

$$p(E) = \frac{1}{Z(\beta)} \rho(E) e^{-\beta E}. \quad (1)$$

$e^{-\beta E}$ is the Boltzmann factor, $\rho(E)$ is the density of states and $Z(\beta)$ is the normalization constant of $\rho(E)e^{-\beta E}$. For superstatistical systems, one generalizes this approach by assuming that β is a random variable as well. Indeed, a driven nonequilibrium system is often inhomogeneous and consist of many spatial cells with different values of β in each cell. The cell size is effectively given by the correlation length of the continuously varying β -field. If we assume that each cell reaches local equilibrium very fast, i.e. the associated relaxation time is short, then in the long-term run the stationary probability distributions $p(E)$ arise as the following mixture of Boltzmann factors:

$$p(E) = \int_0^\infty f(\beta) \frac{1}{Z(\beta)} \rho(E) e^{-\beta E} d\beta \quad (2)$$

Without restriction of generality, we may absorb the factor $1/Z(\beta)$ into the function $f(\beta)$, i.e. define $\tilde{f}(\beta) = f(\beta)/Z(\beta)$ and rename $\tilde{f} \rightarrow f$. Also, for reasons of simplicity we may just assume $\rho(E) = 1$, keeping in mind that the most general case may correspond to a different density of states. The result is an effective distribution

$$p(E) \sim \int_0^\infty f(\beta) e^{-\beta E} d\beta \quad (3)$$

given essentially by the Laplace transform of $f(\beta)$.

The simplest dynamical example of a superstatistical system is a Brownian particle of mass m moving through a changing environment in d dimensions. For its velocity \vec{v} one has the local Langevin equation

$$\dot{\vec{v}} = -\gamma \vec{v} + \sigma \vec{L}(t) \quad (4)$$

($\vec{L}(t)$: d -dimensional Gaussian white noise, γ : friction constant, σ : strength of noise) which becomes superstatistical if the parameter $\beta := \frac{2}{m} \frac{\gamma}{\sigma^2}$ is regarded as a random variable as well. In a fluctuating environment β may vary from cell to cell on a large spatio-temporal scale. Of course, for this example the energy E is just kinetic energy $E = \frac{1}{2}m\vec{v}^2$. In each cell of constant β the local stationary velocity distribution is Gaussian,

$$p(\vec{v}|\beta) = \left(\frac{\beta}{2\pi}\right)^{d/2} e^{-\frac{1}{2}\beta m\vec{v}^2}, \quad (5)$$

provided the relaxation time γ^{-1} is small enough as compared to the changes of β . The marginal distribution describing the long-time behaviour of the particle

$$p(\vec{v}) = \int_0^\infty f(\beta)p(\vec{v}|\beta)d\beta \quad (6)$$

exhibits fat tails. The large- $|v|$ tails of the distribution (6) depend on the behaviour of $f(\beta)$ for $\beta \rightarrow 0$ [8]. Different superstatistical models corresponding to different $f(\beta)$: The function f is determined by the environmental dynamics of the nonequilibrium system under consideration.

Consider, for example, a simple case where there are n independent Gaussian random variables X_1, \dots, X_n underlying the dynamics of β in an additive way. β needs to be positive and a positive β is obtained by squaring these Gaussian random variables. The probability distribution of a random variable that is the sum of squared Gaussian random variables $\beta = \sum_{i=1}^n X_i^2$ is well known in statistics: It is the χ^2 -distribution of degree n , i.e. the probability density $f(\beta)$ is given by

$$f(\beta) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{n}{2\beta_0}\right)^{n/2} \beta^{n/2-1} e^{-\frac{n\beta}{2\beta_0}}, \quad (7)$$

where β_0 is the average of β . In this case the integration over the fluctuating β can be explicitly done and one obtains [12]

$$p(\vec{v}) \sim \frac{1}{(1 + \tilde{\beta}(q-1)\frac{1}{2}m\vec{v}^2)^{\frac{1}{q-1}}} \quad (8)$$

with

$$q = 1 + \frac{2}{n+d} \quad (9)$$

and

$$\tilde{\beta} = \frac{2\beta_0}{2 - (q-1)d}. \quad (10)$$

These types of distributions, so-called q -exponentials, are relevant for nonextensive statistical mechanics [1, 2, 3, 4].

For other systems the random variable β may be generated by multiplicative random processes. In such cases one typically ends up with a log-normally distributed β , i.e. the probability density is given by

$$f(\beta) = \frac{1}{\sqrt{2\pi}s\beta} \exp \left\{ \frac{-(\ln \frac{\beta}{\mu})^2}{2s^2} \right\}, \quad (11)$$

where μ and s are parameters. The corresponding superstatistics is relevant for turbulent systems [28, 29, 30]. The integration over β cannot be done analytically in this case.

3 Generalization

Let us now generalize the additive case. For more general nonequilibrium situations, it may not β itself that is χ^2 -distributed, but some suitable power of β , say $y := \beta^\eta$, where η is a parameter. What the relevant power η is may depend on the physical nature of the nonequilibrium system under consideration and the physical meaning of β . η can depend on the spatial dimensions of the experiment under consideration, its boundary conditions, the dissipation mechanism, the nature of the driving forces, etc.

To construct a microscopic model for $y = \beta^\eta$, we may assume that there are many (nearly) independent microscopic random variables ξ_j , $j = 1, \dots, J$, contributing to y in an additive way. For large J their rescaled sum $\frac{1}{\sqrt{J}} \sum_{j=1}^J \xi_j$ will approach a Gaussian random variable X_1 due to the Central Limit Theorem. In total, there can be many different random variables consisting of microscopic random variables, i.e., we have n Gaussian random variables X_1, \dots, X_n due to various relevant degrees of freedom in the system. y must be positive. A positive value is obtained by squaring the X_i . By construction, the resulting $y = \sum_{i=1}^n X_i^2$ is χ^2 -distributed with degree n , i.e. one has the probability density

$$f_y(y) = \frac{1}{\Gamma(\frac{n}{2})} \left(\frac{n}{2y_0} \right)^{n/2} y^{n/2-1} e^{-\frac{ny}{2y_0}}, \quad (12)$$

where y_0 is the average of y .

A priori, all kinds of values of η are possible. For example, $\eta = -1$ means that the temperature β^{-1} is the relevant quantity that can be represented as a sum of several squared Gaussian random variables. Another case may be that the local standard deviation of the Gaussians is χ^2 -distributed, corresponding to the case $\eta = -\frac{1}{2}$, and so on. Each choice of η implies a different probability distribution of β . The resulting probability density $f_\beta(\beta)$ can simply be obtained by employing a transformation from the χ^2 -distributed random variable $y = \beta^\eta$ to the original random variable β via $f_y(y)dy = f_\beta(\beta)d\beta$. One obtains

$$f_\beta(\beta) = \frac{|\eta|}{\Gamma(\frac{n}{2})} \left(\frac{n}{2y_0}\right)^{\frac{n}{2}} \beta^{\eta\frac{n}{2}-1} e^{-\frac{n}{2y_0}\beta^\eta}, \quad (13)$$

where $y_0 = \langle y \rangle = \langle \beta^\eta \rangle$ is the average of y . As we shall see in the next section, for $\eta > 0$ this generates asymptotic power laws for $p(E)$, whereas the case $\eta < 0$ generates stretched exponentials. Both cases can be physically relevant.

4 Asymptotic behaviour

Let us study the marginal probability distributions $p(E)$ as given by eq. (3) for our class of inverse temperature distributions (13). In particular, we are interested in the behaviour of $p(E)$ for large E , i.e. the shape of the tails.

Two important cases have to be distinguished, $\eta > 0$ and $\eta < 0$. As has been shown in [8], the large-energy tails of $p(E)$ are determined by the behaviour of the function $f(\beta)$ for $\beta \rightarrow 0$. For $\eta > 0$ eq. (13) yields the small- β asymptotics

$$f_\beta(\beta) \sim \beta^{\eta\frac{n}{2}-1}, \quad (14)$$

since the exponential term $\exp(-\frac{n}{2y_0}\beta^\eta)$ in eq. (13) just approaches 1. According to the general results of [8] this implies

$$p(E) \sim E^{-\eta\frac{n}{2}} \quad (15)$$

for $E \rightarrow \infty$. This means one obtains an asymptotic power-law decay in E with the exponent $\eta\frac{n}{2}$. For example, if the energy is just kinetic energy, $E = \frac{1}{2}mv^2$, then $p(v)$ decays with tails proportional to $|v|^{-\eta n}$. This reminds

us of Tsallis statistics and nonextensive statistical mechanics [1, 2, 3, 4], with an entropic index q given by

$$\frac{2}{q-1} = \eta n \Leftrightarrow q = 1 + \frac{2}{\eta n}. \quad (16)$$

However, only $\eta = 1$ yields exact Tsallis statistics, whereas the $p(E)$ obtained for other $\eta > 0$ behave slightly different for finite values of E . Asymptotically, however, only the product of the two parameters η and n is relevant. The same asymptotic power law can be achieved by e.g. doubling the number of degrees of freedom n if at the same time the exponent η is reduced by a factor $1/2$.

The case $\eta < 0$ is very different. Here for small β the behaviour of the function $f(\beta)$ is dominated by the exponential term

$$f(\beta) \sim e^{-c\beta^n}. \quad (17)$$

According to the general results of [8], this implies the large energy behaviour

$$p(E) \sim e^{aE^{\frac{\eta}{\eta-1}}}, \quad (18)$$

where a is a negative constant¹ depending on $c = \frac{n}{2y_0}$ and η :

$$a = -\frac{1 + \frac{1}{|\eta|}}{(c|\eta|)^{\frac{1}{\eta-1}}} \quad (19)$$

We see that $p(E)$ now asymptotically decays as a stretched exponential. It is interesting to see that the degrees of freedom n do not influence the exponent of the stretched exponential, they only influence the proportionality constant a .

5 Applications

For granular gases, stretched exponential tails (and tales!) are known to play an important role [43, 44, 45]. For example, Rouyer and Menon [44] performed an experiment where they measured the probability distribution of velocities in a granular system confined to a vertical plane and driven

¹There is a misprint in eq. (29) of [8], which is hereby corrected.

by strong vertical vibration. The system reaches a stationary state with a probability density of horizontal velocities given by

$$p(v) \sim e^{-c|v|^\alpha}, \quad (20)$$

where α is measured to be $\alpha = 1.55 \pm 0.1$. This result is robust for all frequencies and amplitudes. The theoretical explanation of this behaviour is still unclear.

We may assume that a superstatistics of the type described in this paper is relevant. That means we assume that some suitable power $y = \beta^\eta$ of the inverse granular temperature β is χ^2 -distributed due to large-scale spatio-temporal inhomogeneities of granular temperature. We obtain agreement with the experimentally observed stretched exponential tails if η is given by

$$2\frac{\eta}{\eta-1} = \alpha \Leftrightarrow \eta = \frac{\alpha}{\alpha-2} \approx -3.4. \quad (21)$$

Of course a complete theory would have to explain why in this experiment granular temperature raised to a power of approximately 3.4 is the relevant χ^2 -distributed observable. Currently this is out of reach. The value $\eta = -3.4$ seems not to be universal. Molecular dynamics simulations [45] of dilute granular gases driven by vertically oscillating plates yield exponents α in the range $1.2 < \alpha < 1.6$, which is equivalent to $-4.0 < \eta < -2.3$.

Stretched exponentials are sometimes also used to fit data in hydrodynamic turbulence, despite a lack of theory for this. For example, Bodenschatz et al. [26] present a fit of the measured probability distributions $p(a)$ of acceleration a of a Lagrangian tracer particle which asymptotically decays as a stretched exponential,

$$p(a) \sim e^{-c|a|^{0.4}}. \quad (22)$$

This behaviour, interpreted in terms of a superstatistics, would correspond to $\eta = -1/4$. This would mean that the square root of the standard deviation of the local Gaussians would be the relevant χ^2 -distributed random variable.

Overall, one has to be careful with over-ambitious fits of experimental data using stretched exponentials. Power laws as given by q -exponentials, lognormal superstatistics and stretched exponentials can all yield very similar looking fits of fat-tailed distributions if the largest accessible energy E lies within 5-10 standard deviations. In typical experimental situations the asymptotics of the tails is often not reached. So there can be many competing models to explain the data.

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