

# Superstatistical turbulence models

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## Abstract

Recently there has been some progress in modeling the statistical properties of turbulent flows using simple superstatistical models. Here we briefly review the concept of superstatistics in turbulence. In particular, we discuss a superstatistical extension of the Sawford model and compare with experimental data.

Turbulence is a spatio-temporal chaotic dynamics generated by the Navier-Stokes equation

$$\dot{\vec{v}} = -(\vec{v}\nabla)\vec{v} + \nu\Delta\vec{v} + \vec{F}. \quad (1)$$

In the past 5 years there has been some experimental progress in Lagrangian turbulence measurements, i.e. tracking single tracer particles in the turbulent flow. Due to the measurements of the Bodenschatz [1, 2, 3] and Pinton groups [4, 5] we now have a better view of what the statistics of a single test particle in a turbulent flow looks like. The recent measurements have shown that the acceleration  $\vec{a}$  as well as velocity difference  $\vec{u} = \vec{v}(t + \tau) - \vec{v}(t)$  on short time scales  $\tau$  exhibits strongly non-Gaussian behavior. This is true for both, single components as well as the absolute value of  $\vec{a}$  and  $\vec{u}$ . Moreover, there are correlations between the various components of  $\vec{a}$ , as well as between velocity and acceleration. The corresponding joint probabilities do not factorize. Finally, the correlation functions of the absolute value  $|\vec{a}|$  and  $|\vec{u}|$  decay rather slowly.

How can we understand all this by simple stochastic models? There is a recent class of models that are pretty successful in explaining all these statistical properties of Lagrangian turbulence (as well as of other turbulent systems, such as Eulerian turbulence [6, 7, 8], atmospheric turbulence [9, 10, 11] and defect turbulence [12]). These are turbulence models based on superstatistics [13]. Superstatistics is a concept from nonequilibrium statistical mechanics, in short it means a ‘statistics of statistics’, one given by ordinary Boltzmann factors and another one given by fluctuations of an intensive parameter, e.g. the inverse temperature, or the energy dissipation, or a local variance. While the idea of fluctuating intensive parameters is certainly not new, it is the application to spatio-temporally chaotic systems such as turbulent flow that makes the concept interesting. The first turbulence model of this kind was introduced in [14], in the meantime the idea has been further refined and extended [15, 16, 3, 8]. The basic idea is to generate a superposition of two statistics, in short a ‘superstatistics’, by stochastic differential equations whose parameters fluctuate on a relatively large spatio-temporal scale. In Lagrangian turbulence, this large time scale can be understood by the fact that the particle is trapped in vortex tubes for quite a while [3]. Superstatistical turbulence models reproduce all the experimental data quite well. An example is shown in Fig. 1. The

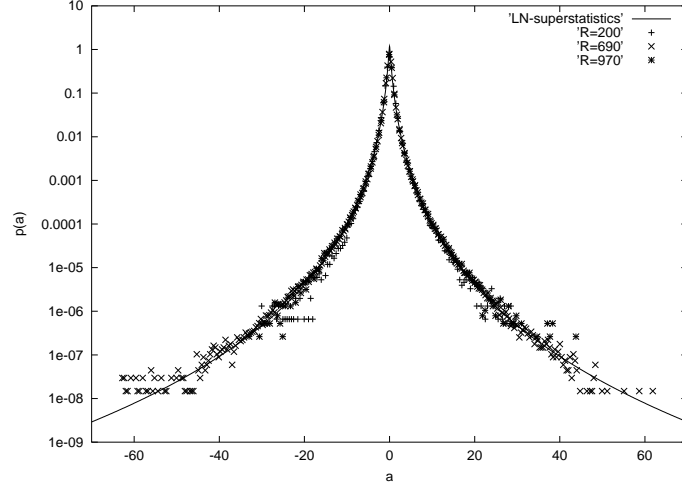


Figure 1: Probability density of an acceleration component of a tracer particle as measured by Bodenschatz et al. [1, 2]. The solid line is a theoretical prediction based on lognormal superstatistics ( $s^2 = 3$ ) [16].

theoretical prediction which fits the data perfectly is given by

$$p(a) = \frac{1}{2\pi s} \int_0^\infty d\beta \beta^{-1/2} \exp \left\{ -\frac{(\log(\beta/\mu))^2}{2s^2} \right\} e^{-\frac{1}{2}\beta a^2} \quad (2)$$

with  $\mu = e^{\frac{1}{2}s^2}$  and only one fitting parameter,  $s^2 = 3.0$ . A similar formula as eq. (2) was already considered in [17], though without a dynamical interpretation in terms of a stochastic differential equation with fluctuating parameters.

The key ingredient of superstatistical models is to start from a known model generating Gaussian behaviour, and extend it to a superstatistical version exhibiting ‘fat tails’. In general, in these types of models one has for some dynamical variable  $a$  the stationary long-term density

$$p(a) = \int_0^\infty \sqrt{\frac{\beta}{2\pi}} f(\beta) e^{-\frac{1}{2}\beta a^2} d\beta, \quad (3)$$

where  $f(\beta)$  is some suitable probability density of a fluctuating parameter  $\beta$ . The function  $f(\beta)$  fixes the type of superstatistics under consideration. In particular, it is responsible for the shape of the tails [18]. Note the mixing of two statistics, that of  $a$  and that of  $\beta$ .

In Lagrangian turbulence, one may first start from a Gaussian turbulence model, the Sawford model [19, 20]. This model considers the joint stochastic process  $(a(t), v(t), x(t))$  of an arbitrary component of acceleration, velocity and position of a Lagrangian test particle, and assumes that they obey the stochastic differential equation

$$\dot{a} = -(T_L^{-1} + t_\eta^{-1})a - T_L^{-1}t_\eta^{-1}v + \sqrt{2\sigma_v^2(T_L^{-1} + t_\eta^{-1})T_L^{-1}t_\eta^{-1}} L(t) \quad (4)$$

$$\dot{v} = a \quad (5)$$

$$\dot{x} = v, \quad (6)$$

$L(t)$ : Gaussian white noise

$T_L$  and  $t_\eta$ : two time scales, with  $T_L \gg t_\eta$ ,

$$T_L = 2\sigma_v^2 / (C_0 \bar{\epsilon})$$

$$t_\eta = 2a_0 \nu^{1/2} / (C_0 \bar{\epsilon}^{1/2})$$

$\bar{\epsilon}$ : average energy dissipation

$C_0, a_0$ : Lagrangian structure function constants

$\sigma_v^2$  variance of the velocity distribution

$R_\lambda = \sqrt{15} \sigma_v^2 / \sqrt{\nu \bar{\epsilon}}$  Taylor scale Reynolds number.

For our purposes it is sufficient to consider the limit  $T_L \rightarrow \infty$ , which is a good approximation for large Reynolds numbers. In that limit the Sawford model reduces to just a linear Langevin equation

$$\dot{a} = -\gamma a + \sigma L(t) \quad (7)$$

with

$$\gamma = \frac{C_0}{2a_0} \nu^{-1/2} \bar{\epsilon}^{1/2} \quad (8)$$

$$\sigma = \frac{C_0^{3/2}}{2a_0} \nu^{-1/2} \bar{\epsilon}. \quad (9)$$

Note that this is a Langevin equation for the acceleration, not for the velocity, in marked contrast to ordinary Brownian motion.

Unfortunately, the Sawford model predicts Gaussian stationary distributions for  $a$ , and is thus at variance with the recent measurements. So how can we save this model?

As said before, the idea is to generalize the Sawford model with constant parameters to a superstatistical version. To construct a superstatistical extension of Sawford model, we replace in the above equations the constant energy dissipation  $\bar{\epsilon}$  by a fluctuating one. One formally defines a variance parameter [16]

$$\beta := \frac{2\gamma}{\sigma^2} = \frac{4a_0}{C_0^2} \nu^{1/2} \frac{1}{\epsilon^{3/2}}, \quad (10)$$

where  $\epsilon$  fluctuates. Now, if  $\beta$  varies on a large spatio-temporal scale, and is distributed with the distribution  $f(\beta)$ , one ends up with eq. (3) describing the long-term marginal distribution of the superstatistical dynamics (7). This is basically the type of model introduced in [14], there with  $f(\beta)$  chosen to be a  $\chi^2$ -distribution. Models based on  $\chi^2$ -superstatistics yield good results for atmospheric turbulence [9, 10], and ultimately lead to Tsallis statistics [21]. On the other hand, for laboratory turbulence experiments one usually obtains better agreement with experimental data if  $f(\beta)$  is a lognormal distribution. In view of eq. (10) this is clearly motivated by Kolmogorov's ideas of a lognormally distributed  $\epsilon$  [22].

Superstatistical models are not restricted to Lagrangian turbulence but can be also formulated for Eulerian turbulence [7, 14]. Fig. 2 shows that also here one obtains excellent agreement with experimental data: Probability densities  $p(u)$  of longitudinal velocity differences  $u$  are well fitted by lognormal superstatistics on all scales. The parameter  $s^2$  varies with the scale. In fact, not only the distribution  $p(u)$  but also the distribution  $f(\beta)$  can be directly measured in experiments [8], and the two can be consistently connected via the superstatistics formalism. Jung and Swinney [8] have also experimentally confirmed a simple scaling relation between  $\beta$  and the fluctuating energy dissipation  $\epsilon$ .

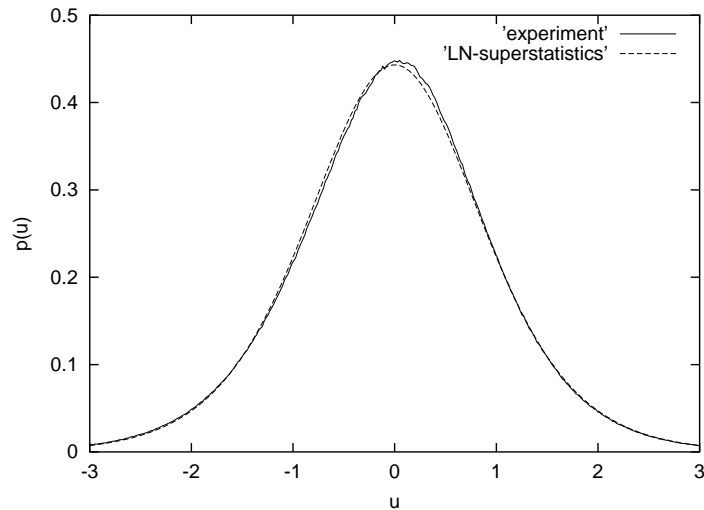


Figure 2: Experimentally measured histogram of velocity differences in a Taylor-Couette experiment [6], and comparison with a superstatistical prediction ( $s^2 = 0.28$ ).

It should be noted that if we know the probability densities  $p(u)$  analytically, as well as the dependence of the parameter  $s^2$  on the scale  $r$ , we can also calculate moments of velocity differences and thus determine scaling exponents  $\zeta_m$  defined by

$$\langle u^m \rangle \sim r^{\zeta_m}.$$

Many different models of  $\zeta_m$  can be constructed in such a way [16, 23, 24]. For further stochastic models, see e.g. [25, 26, 27].

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