Time Series Course-work 4
Solutions to the Theory Questions

Question 1
To derive confidence intervals for $\rho(\tau)$ we may use Bartlett's formula for variance of the estimator of $\rho(\tau)$ and the fact that the estimator is approximately normally distributed, that is

$$\widehat{\boldsymbol{\rho}} \underset{approx}{\sim} \mathcal{N}\left(\boldsymbol{\rho}, \frac{1}{n}\boldsymbol{W}\right),$$

where

$$\boldsymbol{\rho} = (\rho(1), \ldots, \rho(k))^{\mathrm{T}}$$

and $\boldsymbol{W}$ is the variance-covariance matrix

$$\boldsymbol{W} = \{w_{ij}\},$$

where w_{ij} is given by Bartlett's formula

$$w_{ij} = \sum_{k=1}^{\infty}[\rho(k+i) + \rho(k-i) - 2\rho(i)\rho(k)][\rho(k+j) + \rho(k-j) - 2\rho(j)\rho(k)].$$

Then, a 95% confidence interval is

$$\widehat{\rho(\tau)} \pm 1.96\sqrt{\frac{w_{\tau\tau}}{n}}.$$

For an MA(2) model

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, \quad \text{where } Z_t \sim WN(0, \sigma^2)$$

we have non-zero auto-correlations $\rho(\tau)$ for $\tau = 0, \pm1, \pm2$ and zero auto-correlations for all other values of lag τ.

For $\tau = 1$ and $\tau = 2$ we obtain, respectively

$$\begin{aligned} w_{11} &= [\rho(2) + 1 - 2\rho^2(1)]^2 + [\rho(1) - 2\rho(1)\rho(2)]^2 + \rho^2(2) \\ w_{22} &= [\rho(1) - 2\rho(1)\rho(2)]^2 + [1 - 2\rho^2(2)]^2 + \rho^2(1) + \rho^2(2) \end{aligned}$$

For all other lags, that is for $|\tau| > 2$ we obtain

$$w_{\tau\tau} = 2\rho^2(2) + 2\rho^2(1) + 1.$$

We know that the ACF for $|\tau| > 2$ is zero for an MA(2), hence the CIs at these lags can be used for testing non-significance of the auto-correlations, that is checking the model fit.

Question 2
Given $\widehat{\rho(1)} = 0.5357$, $\widehat{\rho(2)} = 0.3961$ and $n = 84$, we obtain the following variances of the estimators of the ACF of the MA(2) process:

$$w_{\tau\tau} = \begin{cases} 0.84522 & \text{for} \quad |\tau| = 1 \\ 0.92715 & \text{for} \quad |\tau| = 2 \\ 1.88774 & \text{for} \quad |\tau| > 2 \end{cases}$$

These give the following 95% confidence intervals for $\rho(\tau)$:

$$\begin{cases} (0.339092, 0.732308) & \text{for} \quad |\tau| = 1 \\ (0.190184, 0.602016) & \text{for} \quad |\tau| = 2 \\ (-0.293824, 0.293824) & \text{for} \quad |\tau| > 2 \end{cases}$$