# Time Series Course-work 1
# Solutions to the Theory Questions

**Question 1**
See the derivation of the linear filter given in the lecture notes, Chapter 2, pages 18-21.

**Question 2**
This moving average is based on the expansion of the expression $(a+b)^n$:

$$(a+b)^n = \sum_{r=0}^{n} \binom{n}{r} a^{n-r} b^r.$$

The successive terms in the expansion of $\left(\frac{1}{2}+\frac{1}{2}\right)^n$ are

$$a^{n-r}b^r = \left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{2}\right)^{r} = \left(\frac{1}{2}\right)^{n} \quad \text{for } r = 0, 1, ..., n.$$

For $q = 2$ we have $n = 2q = 4$ and

$$a^{n-r}b^r = \left(\frac{1}{2}\right)^{4} = \frac{1}{16} \quad \text{for } r = 0, 1, 2, 3, 4.$$

The coefficients $\binom{n}{r}$ are

$$\binom{4}{0} = 1, \ \binom{4}{1} = 4, \ \binom{4}{2} = 6, \ \binom{4}{3} = 4, \ \binom{4}{4} = 1.$$

Hence, the m.a. is

$$\{a_j\} = \frac{1}{16}(1, 4, \mathbf{6}).$$

For $q = 3$ we obtain

$$\{a_j\} = \frac{1}{64}(1, 6, 15, \mathbf{20}).$$

**Question 3**
See lecture notes, pages 22-23. Convolution of two linear filters is a filter which is obtained as

$$\{c_i\} = \{a_j\} \star \{b_k\},$$

where $\{c_i\}$ is the sum of all products for which $j + k = i$. Here we have

$$\begin{aligned}\{c_i\} =& (a_{-2}, a_{-1}, a_0, a_1) \star (b_0, b_1, b_2)\\ =& (a_{-2}b_0, a_{-2}b_1 + a_{-1}b_0, a_{-2}b_2 + a_{-1}b_1 + a_0b_0, a_{-1}b_2 + a_0b_1 + a_1b_0, a_0b_2 + a_1b_1, a_1b_2)\\ =& (c_{-2}, c_{-1}, c_0, c_1, c_2, c_3).\end{aligned}$$

Hence

$$\left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right) \star \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$
$$\left(\frac{1}{12}, \frac{1}{6}, \frac{1}{4}, \frac{1}{4}, \frac{1}{6}, \frac{1}{12}\right)$$

and the new series $Z_t$ is

$$Z_t = \frac{1}{12}X_{t-2} + \frac{1}{6}X_{t-1} + \frac{1}{4}X_t + \frac{1}{4}X_{t+1} + \frac{1}{6}X_{t+2} + \frac{1}{12}X_{t+3}.$$

**Question 4**

$$\begin{aligned}\nabla^3 X_t =& \nabla(\nabla^2 X_t) = (1-B)(1-2B+B^2)X_t\\ =& (1-3B+3B^2-B^3)X_t = X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3}.\end{aligned}$$

Now,

$$(a_{-1}, a_0) \star (b_{-1}, b_0) = (a_{-1}b_{-1}, a_{-1}b_0 + a_0 b_{-1}, a_0 b_0) = (c_{-2}, c_{-1}, c_0).$$

Hence,

$$(-1, 1) \star (-1, 1) = (1, -1-1, 1) = (1, -2, 1).$$

Then,

$$(c_{-2}, c_{-1}, c_0) \star (d_{-1}, d_0) = (c_{-2}d_{-1}, c_{-2}d_0 + c_{-1}d_{-1}, c_{-1}d_0 + c_0 d_{-1}, c_0 d_0)$$

and so

$$(1, -2, 1) \star (-1, 1) = (-1, 1+2, -2-1, 1) = (-1, 3, -3, 1)$$

what gives the following linear filter

$$X_t - 3X_{t-1} + 3X_{t-2} - X_{t-3},$$

which is equal to $\nabla^3 X_t$.