4.2 Strict Stationarity

A more restrictive definition of stationarity involves all the multivariate distributions of the subsets of TS r.vs.

Definition 4.4. *A time series $\{X_t\}$ is called* ***strictly stationary*** *if the random vectors $(X_{t_1}, \dots, X_{t_n})^{\mathrm{T}}$ and $(X_{t_1+\tau}, \dots, X_{t_n+\tau})^{\mathrm{T}}$ have the same joint distribution for all sets of indices $\{t_1, \dots, t_n\}$ and for all integers τ and $n > 0$. It is written as*

$$(X_{t_1}, \dots, X_{t_n})^{\mathrm{T}} \stackrel{d}{=} (X_{t_1+\tau}, \dots, X_{t_n+\tau})^{\mathrm{T}},$$

where $\stackrel{d}{=}$ means "equal in distribution".

Properties of a Strictly Stationary TS

1. *The r.vs X_t are identically distributed for all t.*
   Follows from the definition for $n = 1$.

2. *Pairs of r.vs $(X_t, X_{t+\tau})^{\mathrm{T}}$ are identically distributed for all t and τ. That is,*

   $$(X_t, X_{t+\tau})^{\mathrm{T}} \stackrel{d}{=} (X_1, X_{1+\tau})^{\mathrm{T}}.$$

   Follows from the definition for $n = 2$.

3. *X_t is a weakly stationary TS if $\mathrm{E}(X_t^2) < \infty$ for all t.*
   By (1) $\mathrm{E}\,X_t$ is constant, does not depend on t. If $\mathrm{E}(X_t^2) < \infty$ then all covariances also exist (by Cauchy-Schwarz inequality). By (2) $\gamma(\tau) = \mathrm{cov}(X_t, X_{t+\tau}) = \mathrm{cov}(X_1, X_{1+\tau})$, which also does not depend on t.

4. *Weak stationarity does not imply strict stationarity.*
   To prove this property we show an example of a weakly stationary TS which is not strictly stationary. Let $Z_t \underset{iid}{\sim} \mathcal{N}(0, 1)$. Define

   $$X_t = \begin{cases} Z_t, & \text{if } t \text{ is even,} \\ \frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1) & \text{if } t \text{ is odd.} \end{cases}$$

   Then

   $$\mathrm{E}\,X_t = \begin{cases} \mathrm{E}\,Z_t = 0, & \text{if } t \text{ is even,} \\ \mathrm{E}\left[\frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1)\right] = \frac{1}{\sqrt{2}}\,\mathrm{E}[Z_{t-1}^2 - 1] = 0 & \text{if } t \text{ is odd.} \end{cases}$$

   Also,

   $$\mathrm{var}(X_t) = \begin{cases} \mathrm{var}(Z_t) = 1, & \text{if } t \text{ is even,} \\ \mathrm{var}(\frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1)) = \frac{1}{2}\,\mathrm{var}(Z_{t-1}^2) = 1 & \text{if } t \text{ is odd,} \end{cases}$$

and by part 2 of Theorem 3.2 we obtain

$$\operatorname{cov}(X_t, X_{t+\tau}) = 0.$$

Hence X_t is a weakly stationary TS, it is in fact $WN(0, 1)$. Is it identically distributed? We will compare $P(X_t < x_t)$ for t even and odd. We have

$$P(X_t < x_t) = P(Z_t < x_t) \quad \text{for } t \text{ even}.$$

For t odd we get

$$\begin{aligned} P(X_t < x_t) &= P\left(\frac{1}{\sqrt{2}}(Z_{t-1}^2 - 1) < x_t\right) \\ &= P\left(Z_{t-1}^2 < \sqrt{2}x_t + 1\right) \\ &= P\left(-\sqrt{\sqrt{2}x_t + 1} < Z_{t-1} < \sqrt{\sqrt{2}x_t + 1}\right). \end{aligned}$$

Take $x = 0$. Then

$$P(X_t < 0) = \begin{cases} 0.5, & \text{if } t \text{ is even}, \\ 0.6826, & \text{if } t \text{ is odd}. \end{cases}$$

Hence the c.d.fs are different for t even and t odd, that is series X_t is not strictly stationary.

5. *An i.i.d. TS is strictly stationary*.
   For an i.i.d. TS the joint cdf is

$$P(X_{t_1+\tau} < x_1, \ldots, X_{t_n+\tau} < x_n) = F(x_1) \cdot \ldots \cdot F(x_n).$$

So, it does not depend on the choice of the indices $\{t_1, \ldots, t_n\}$.